ASME PTC 19.1-2018
(Revision of ASME PTC 19.1-2013)

Test Uncertainty

Performance Test Codes

AN AMERICAN NATIONAL STANDARD

The American Society of
Mechanical Engineers

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The American Society of Mechanical Engineers
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The American Society of Mechanical Engineers
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NOTICE

All Performance Test Codes must adhere to the requirements of ASME PTC 1, General Instructions. The following information is based on that document and is included here for emphasis and for the convenience of the user of the Supplement. It is expected that the Code user is fully cognizant of Sections 1 and 3 of ASME PTC 1 and has read them prior to applying this Supplement.

ASME Performance Test Codes provide test procedures that yield results of the highest level of accuracy consistent with the best engineering knowledge and practice currently available. They were developed by balanced committees representing all concerned interests and specify procedures, instrumentation, equipment-operating requirements, calculation methods, and uncertainty analysis.

When tests are run in accordance with a code, the test results themselves, without adjustment for uncertainty, yield the best available indication of the actual performance of the tested equipment. ASME Performance Test Codes do not specify means to compare those results with contractual guarantees. Therefore, it is recommended that the parties to a commercial test agree before starting the test and preferably before signing the contract on the method to be used for comparing the test results with the contractual guarantees. It is beyond the scope of any code to determine or interpret how such comparisons shall be made.

FOREWORD

In March 1979, the Performance Test Codes Supervisory Committee activated the PTC 19.1 Committee to revise a 1969 draft of the document PTC 19.1, General Considerations. The PTC 19.1 Committee proceeded to develop a Performance Test Code Instruments and Apparatus Supplement published in 1985 as PTC 19.1-1985, Measurement Uncertainty. This, along with its subsequent editions, was intended to provide a means to standardize nomenclature, symbols, and methodology of measurement uncertainty in ASME Performance Test Codes.

Work on the revision of the original 1985 edition began in 1991 with the two-fold objective of improving its usefulness to the reader through greater clarity, conciseness, and technical treatment of the evolving subject matter; and harmonizing with ISO/IEC Guide 98-3, Guide to the Expression of Uncertainty in Measurement (GUM). ASME published PTC 19.1-1998 as Test Uncertainty, the new title reflecting the appropriate orientation of the document.

The effort to update the 1998 revision began immediately upon completion of that document. The 2005 revision was notable for the following significant departures from the 1998 text:

(a) ASME PTC 19.1-2005 adopted nomenclature more consistent with ISO/IEC Guide 98-3. Uncertainties remained conceptualized as "systematic" (estimate of the effects of fixed error not observed in the data) and "random" (estimate of the limits of the error observed from the scatter of the test data). Both types of uncertainty were defined at the standard-deviation level as "standard uncertainties." The determination of an uncertainty at some level of confidence was based on the root-sum-square of the systematic and random standard uncertainties multiplied by the appropriate expansion factor for the desired level of confidence (usually "2" for 95%). This same approach was used in the 1998 revision, but the characterization of uncertainties at the standard-uncertainty level ("standard deviation") was not as explicitly stated. The new nomenclature was expected to render ASME PTC 19.1-2005 and subsequent revisions more acceptable to an international audience.

(b) There was greater discussion of the determination of systematic uncertainties.

(c) Text was added on a simplified approach to determine the uncertainty of straight-line regression.

For this 2018 revision, the significant changes are the addition of the Monte Carlo method for propagating uncertainties and the use of multiple test results to obtain an estimate of the random uncertainty of the result. A detailed example that illustrates all aspects of uncertainty analysis is included as a separate section in the document. This section shows both the Taylor series method and the Monte Carlo method for propagating uncertainties. This new section replaces the examples section that was included in previous versions of the document.

This Standard is available for public review on a continuing basis. This provides an opportunity for additional public-review input from industry, academia, regulatory agencies, and the public-at-large.

ASME PTC 19.1-2018 was approved by the PTC Standards Committee on March 28, 2018, and was approved as an American National Standard by the ANSI Board of Standards Review on September 20, 2018.

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Secretary, PTC Standards Committee
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Proposing Revisions. Revisions are made periodically to the Code to incorporate changes that appear necessary or desirable, as demonstrated by the experience gained from the application of the Code. Approved revisions will be published periodically.

The Committee welcomes proposals for revisions to this Code. Such proposals should be as specific as possible, citing the paragraph number(s), the proposed wording, and a detailed description of the reasons for the proposal, including any pertinent documentation.

Proposing a Case. Cases may be issued to provide alternative rules when justified, to permit early implementation of an approved revision when the need is urgent, or to provide rules not covered by existing provisions. Cases are effective immediately upon ASME approval and shall be posted on the ASME Committee web page.

Requests for Cases shall provide a Statement of Need and Background Information. The request should identify the Code and the paragraph, figure, or table number(s), and be written as a Question and Reply in the same format as existing Cases. Requests for Cases should also indicate the applicable edition(s) of the Code to which the proposed Case applies.

Interpretations. Upon request, the PTC Standards Committee will render an interpretation of any requirement of the Code. Interpretations can only be rendered in response to a written request sent to the Secretary of the PTC Standards Committee.

Requests for interpretation should preferably be submitted through the online Interpretation Submittal Form. The form is accessible at http://go.asme.org/InterpretationRequest. Upon submittal of the form, the Inquirer will receive an automatic e-mail confirming receipt.

If the Inquirer is unable to use the online form, he/she may mail the request to the Secretary of the PTC Standards Committee at the above address. The request for an interpretation should be clear and unambiguous. It is further recommended that the Inquirer submit his/her request in the following format:

Subject: Cite the applicable paragraph number(s) and the topic of the inquiry in one or two words.

Edition: Cite the applicable edition of the Code for which the interpretation is being requested.

Question: Phrase the question as a request for an interpretation of a specific requirement suitable for general understanding and use, not as a request for an approval of a proprietary design or situation. Please provide a condensed and precise question, composed in such a way that a "yes" or "no" reply is acceptable.

Proposed Reply(ies): Provide a proposed reply(ies) in the form of "Yes" or "No," with explanation as needed. If entering replies to more than one question, please number the questions and replies.

Background Information: Provide the Committee with any background information that will assist the Committee in understanding the inquiry. The Inquirer may also include any plans or drawings that are necessary to explain the question; however, they should not contain proprietary names or information.

Requests that are not in the format described above may be rewritten in the appropriate format by the Committee prior to being answered, which may inadvertently change the intent of the original request.

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INTRODUCTION

Most sections in this revision of ASME PTC 19.1-2013 [1] have been rewritten to add to the available technology for uncertainty analysis and to make it easier for the practicing engineer to use. The intent is to provide a standard that can be used easily by engineers and scientists with interest in the objective assessment of measured-parameter data quality using test uncertainty analysis.

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Section 1
Object and Scope

1-1 OBJECT

The object of this Standard is to define, describe, and illustrate the terms and methods used to provide meaningful estimates of the uncertainty in test measurements, parameters, and methods, and the effects of those uncertainties on derived test results.

1-1.1 Objectives

An uncertainty analysis of test measurements, parameters, and methods is useful because it

(a) provides an objective estimate of the quality of test data and results

(b) facilitates communication regarding measurement and test results

(c) fosters an understanding of potential error sources in a measurement system, and the effects of those potential error sources on test results

(d) guides the decision-making process for selecting appropriate and cost-effective measurement systems and methods

(e) reduces the risk of making erroneous decisions based on test results

(f) documents uncertainty for assessing compliance with test requirements

(g) substantiates the test uncertainty budget

When an uncertainty analysis is completed, a numerical characterization of the quality of test results is available with an appropriate level of confidence, typically 95%.

1-2 SCOPE

The scope of this Standard is to specify procedures for

(a) evaluation of uncertainties in test measurements, parameters, and methods

(b) propagation of those uncertainties into the uncertainty of a test result

Depending on the application, uncertainty sources may be classified either by the presumed effect (systematic or random) on the measurement or test result, or by the process in which they may be quantified or their pedigree (Type A or Type B).

1-2.1 Uncertainty Propagation Methods

This Standard incorporates two internationally accepted methods of propagating uncertainties in measured parameters to a derived test result.

1-2.1.1 Taylor Series Method (TSM). This method of propagation is consistent with ISO/IEC Guide 98-3 (GUM) [2]. The TSM requires the determination of sensitivity coefficients for each input variable (how the result is affected by variations in the input variables) and standard uncertainties for each error source.

1-2.1.2 The Monte Carlo Method (MCM). This method of propagation is consistent with JCGM 101 [3]. The MCM requires estimation of probability distributions and standard uncertainties (standard deviations) for each error source.

The distribution determined as the output of an MCM analysis allows direct determination of the lower and upper limits of a coverage interval that contains a specified percentage of the distribution. Thus there are no additional assumptions required to arrive at an "expansion factor," as is necessary in the TSM approach, to obtain a confidence interval estimate.

1-2.2 Uncertainty Propagation Classifications

This Standard uses two major classifications for errors and uncertainties: systematic and random. The ISO GUM uses a different classification for uncertainties: Type A and Type B.

1-2.2.1 Systematic. Systematic errors, whose effects are estimated with "systematic standard uncertainties," do not cause scatter in test data.

1-2.2.2 Random. Random errors, whose effects are estimated with "random standard uncertainties," cause scatter in test data.

1-2.2.3 ISO GUM Classification. The ISO GUM uses a different classification: Type A uncertainties are evaluated with statistical methods and Type B uncertainties are evaluated using other means, such as models or judgment. The terms identify the pedigree of the error sources.

The uncertainty of a test result is independent of whether the elemental uncertainties are classified as systematic or random, or as Type A or Type B. Regardless of the uncertainty classification used, the calculated uncertainty of the result will be the same. While this Standard utilizes systematic and random terms, there may be situations where it is useful to classify elemental uncertainties by effect, source, or both.

1-3 APPLICATIONS

This Standard is intended to serve as a reference to other supplements in the ASME PTC 19 Series and to ASME performance test codes and standards in general. In addition, it is applicable for all known measurement and test uncertainty analyses.

NOTE: The nominal values for the parameters and the uncertainty levels used throughout this Standard are for illustrative purposes only and are not intended to be typical of standard tests. Values and uncertainty levels shall be evaluated for the specific test and measurement system used.

Section 2
Nomenclature and Glossary

2-1 NOMENCLATURE

2-1.1 Symbols

The following symbols are used in this Code:

$B_{\overline{X}_k}$ = 95% confidence level estimate of the limits associated with the k^{th} elemental systematic error source

b_R = systematic standard uncertainty component of a result

$b_{\overline{X}}$ = systematic standard uncertainty component of a measurand

$b_{\overline{X}_k}$ = systematic standard uncertainty associated with the k^{th} elemental error source for a measurand

$b_{\overline{X}_{ns}}$ = systematic standard uncertainty for nonsymmetrical systematic error

b_{XY} = covariance of the systematic errors in X and Y

N = sample size

s_R = random standard uncertainty of a result

s_X = standard deviation of a data sample of a measurand; estimate of the standard deviation of the population σ_x

$s_{\overline{X}}$ = random standard uncertainty of the mean of N observations of a measurand

SEE = standard error of estimate of a least-squares regression or curve fit

t = Student's t value at a specified confidence level with ν degrees of freedom, i.e., $t_{95,\nu}$

U = expanded uncertainty

U^+, U^- = upper and lower values of the nonsymmetrical expanded uncertainty

u = combined standard uncertainty

X = individual observation in a data sample of a measurand

$\overline{X}$ = sample mean; average of a set of N individual observations of a measurand

β = true systematic error (unknown); fixed or constant component of δ

β_k = elemental systematic error

δ = total error (unknown); difference between the assigned value of a parameter or a test result and the true value

ε = true random error (unknown); random component of δ

θ = absolute sensitivity

θ' = relative sensitivity

μ = true average of a population (unknown)

ν = number of degrees of freedom

σ = true standard deviation of a population (unknown)

σ^2 = true variance of a population (unknown)

2-1.2 Indices

I = total number of variables

i = counter for variables

J = total number of sensors

j = counter for individual observations of a measurand

K = total number of sources of elemental errors and uncertainties

k = counter for sources of elemental errors and uncertainties

L = total number of correlated sources of systematic error

l = counter for correlated sources of systematic error

M = total number of multiple results

m = counter for multiple results

N = total number of observations of a measurand

2-2 GLOSSARY

calibration: the process of comparing the response of an instrument to that of a standard instrument over some measurement range.

calibration hierarchy: the established pedigree for a measurement based on the chain of calibrations that links or traces a measuring instrument to a primary standard.

combined standard uncertainty (u): the root-sum-square combination of systematic and random standard uncertainties for a measurement or result.

confidence level: the probability that the true value falls within the specified limits.

degrees of freedom (ν): the number of independent observations used to calculate a statistic.

elemental random error source: an identifiable source of random error that is a subcomponent of total random error.

elemental random standard uncertainty ($s_{\overline{X}_k}$): an estimate of the standard deviation of the mean of the k^{th} elemental random error source.

elemental systematic error source (β_k): an identifiable source of systematic error that is a subcomponent of the total systematic error.

elemental systematic standard uncertainty ($b_{\overline{X}_k}$): a constant value that estimates standard deviation of the k^{th} elemental systematic error source.

error: the difference between the observed value of the measurand and its corresponding true value.

expanded uncertainty ($U_{\overline{X}}$ or U_R): an estimate of the limits of total error, with a defined level of confidence (usually 95%).

influence coefficient: see *sensitivity*.

mean ($\overline{X}$): the arithmetic average of N readings of a measurand.

measurand: the particular quantity that is being measured or estimated.

measurement uncertainty: the uncertainty associated with a measurand. It is an estimate of the expected limits of measurement error.

parameter: a quantity that can be measured from the best available information—such as temperature, pressure, stress, or specific heat—to determine a result. The value used is called the assigned value.

population: the set of all possible values of a parameter.

population mean (μ): the average of the set of all population values of a parameter.

population standard deviation (σ): a value that quantifies the dispersion of a population.

quantity: the property of a phenomenon, body, or substance that has a magnitude that can be expressed as a number.

random error (ε): the portion of total error that varies randomly in repeated measurements of the true value throughout a test process.

random standard uncertainty of the sample mean ($s_{\overline{X}}$): a value that quantifies the dispersion of a sample mean as given by eq. (3-3-3).

result (R): a value calculated from a number of parameters.

sample size (N): the number of observations or values available for a single measurand.

sample standard deviation (s_X): a value that quantifies the dispersion of a sample of measurements as given by eq. (3-3-2). It is an estimate of the standard deviation of the population σ_x.

sensitivity (θ): the rate of change in a result due to a change in a variable evaluated at a desired test operating point.

standard error of estimate (SEE): the measure of dispersion of the dependent variable around a least-squares regression or curve.

statistic: any numerical quantity derived from the sample data. $\overline{X}$ and $s_{\overline{X}}$ are statistics.

Student's t value (t): the coverage factor to calculate expanded uncertainty from the combined standard uncertainty at a specified level of confidence with v degrees of freedom, i.e., $t_{95,v}$

systematic error (β): the portion of total error that remains constant in repeated measurements of the true value throughout a test process. It is a fixed or constant component of δ (unknown).

systematic standard uncertainty ($b_{\overline{X}}$): a value that quantifies the dispersion of a systematic error associated with the mean.

test uncertainty: the uncertainty of a test result.

total error (δ): the unknown difference between the measurement of a parameter or test result and its true value.

traceability: see *calibration hierarchy*.

true value: the unknown, error-free value of a measurand or test result.

Type A uncertainty: a class of uncertainties that use measured data to calculate a standard deviation for use in estimating the uncertainty.

Type B uncertainty: a class of uncertainties that do not use measured data to calculate a standard deviation, thus requiring the uncertainty to be estimated by other methods.

uncertainty: the limits of error within which the true value lies.

uncertainty interval: an interval around a measurand or test result that is expected to contain the true value with a prescribed level of confidence.

variable: a quantity that can be assigned different values that can be measured or counted. It may be calculated from a number of measurands.

Section 3
Fundamental Concepts

3-1 ASSUMPTIONS

The assumptions inherent in test uncertainty analysis include the following:

(a) the test objectives are specified

(b) the test process, including the measurement process and the data reduction process, is defined

(c) the test process, with respect to the conditions of the item under test and the measurement system employed for the test, is controlled for the duration of the test

(d) the measurement system is calibrated and all appropriate calibration corrections are applied to the resulting test data

(e) all appropriate engineering corrections are applied to the test data as part of the data reduction and/or results analysis process

In this Standard, there is a careful distinction between the terms "error" and "uncertainty." Error, discussed in subsection 3-2, is the difference between a particular quantity that is being measured or estimated, called the "measurand," and its corresponding true value. The actual error of a measurement cannot be known but its effect may be estimated. This estimate is called the uncertainty. Uncertainty is an interval around a measurement in which the true value of the measurand is expected to lie.

Uncertainty is not the error of the measurement but an expression of the expected limits for the measurement error at a chosen level of confidence. For expanded uncertainty, 95% level of confidence has been used throughout this document in accordance with accepted practice. Other confidence levels may be used, if required (see Nonmandatory Appendix B).

3-2 MEASUREMENT ERROR

Every measurement has error, which results in a difference between the measured value, X, and the true value. As Figure 3-2-1 illustrates, the difference between the measured value and the true value is the total error, δ. Since the true value is unknown, total error cannot be known and therefore only its expected limits can be estimated. Total error consists of two components: random error and systematic error (see Figure 3-2-1). Reducing measurement error requires reducing random and/or systematic errors. The effect of controlling these error components is highlighted in Figure 3-2-2.

3-2.1 Random Error

Random error, ε, is the portion of the total error that varies in repeated measurements at a set test condition. The total random error in a measurement is the combination of the contributions of several elemental random error sources. Elemental random errors may arise from uncontrolled test conditions and nonrepeatability in the measurement system, measurement methods, environmental conditions, data reduction techniques, etc. Random errors always cause variability (i.e., scatter) in test data.

3-2.2 Systematic Error

Systematic error, β, is the portion of the total error that remains constant in repeated measurements at a set test condition. The total systematic error in a measurement is the sum of the contributions of several elemental systematic errors. Elemental systematic errors may arise from imperfect calibration corrections, measurement methods, environmental conditions, data reduction techniques, etc. Systematic errors are always constant at a set test condition and affect the measurand by the same amount, so their effect cannot be seen in test data.

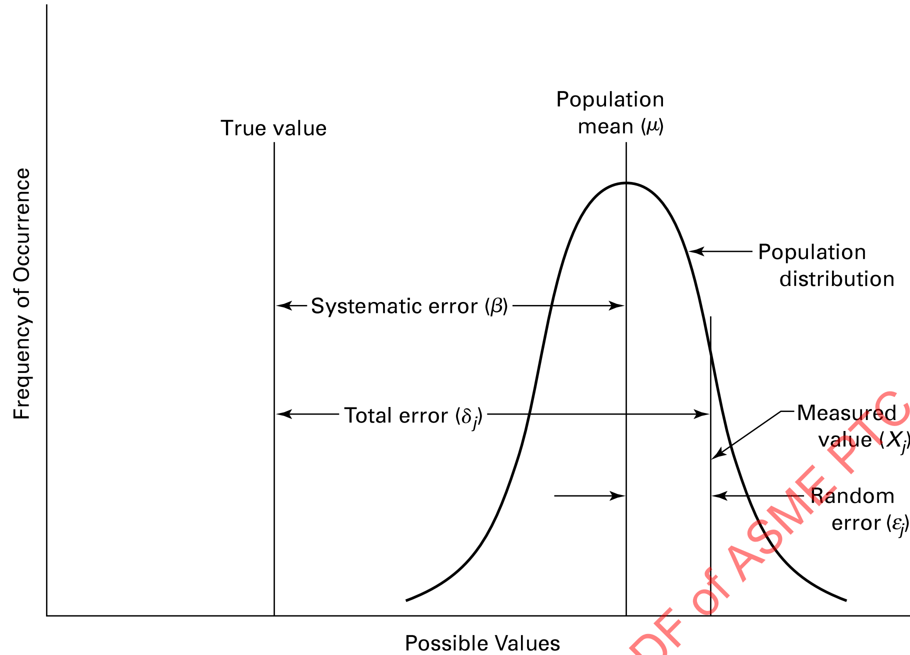
3-3 MEASUREMENT UNCERTAINTY

There is an inherent uncertainty in the use of measurements to represent the true value. Measurement uncertainty refers to the estimated effects of error. The combined uncertainty in a measurement is the combination of uncertainty due to random error and uncertainty due to systematic error. When these uncertainties are evaluated at a standard deviation level, they are called "standard" uncertainties.

3-3.1 Random Standard Uncertainty of a Measurand

Any single measurement of a measurand is influenced by multiple elemental random error sources, ε_j. In successive measurements of a measurand, the values of these elemental random errors change, resulting in the scatter observed in successive measurements. If an

Figure 3-2-1 Illustration of Measurement Errors



infinite number of measurements of a measurand were to be taken with the defined test process, the resulting population of measurements could be described statistically in terms of the population mean, μ, the population standard deviation, σ, and the frequency distribution of the population. These terms are illustrated in Figure 3-3.1-1 for a population of measurements that is normally distributed. For measurements with zero systematic error (see para. 3-2.2), the population mean is equal to the true value of the measurand and the population standard deviation is a measure of the scatter of the individual measurements about the population mean. For a normal distribution, the interval $\mu \pm \sigma$ will include approximately 68% of the population, and the interval $\mu \pm 2\sigma$ will include approximately 95% of the population.

Since at a set test condition only a finite number of measurements are acquired, the population's true mean, μ, and true standard deviation, σ, are unknown but can be estimated from sample statistics. The sample mean, $\overline{X}$, is only an estimate of the population mean and is given by

$$\overline{X} = \frac{\sum_{j=1}^{N} X_j}{N} \tag{3-3-1}$$

where

N = the number of measurements in the sample

X_j = the value of each individual measurement in the sample

The sample standard deviation, s_X, is only an estimate of the population standard deviation and is given by

$$s_X = \sqrt{\sum_{j=1}^{N} \frac{\left(X_j - \overline{X}\right)^2}{N - 1}} \tag{3-3-2}$$

For a distribution of measurements, the standard deviation of the sample mean, $S_{\overline{X}}$, can be used to define the probable interval around the sample mean that is expected to contain the population mean. The standard deviation of the sample mean is related to the sample standard deviation and is called the random standard uncertainty:

$$s_{\overline{X}} = \frac{s_X}{\sqrt{N}} \tag{3-3-3}$$

In general, increasing the number of measurements collected at a set test condition is beneficial because

(a) it improves the sample mean as an estimator of the true population mean

(b) it improves the sample standard deviation as an estimator of the true population standard deviation

(c) it reduces the value of the random standard uncertainty of the sample mean

Figure 3-2-2 Measurement Error Components

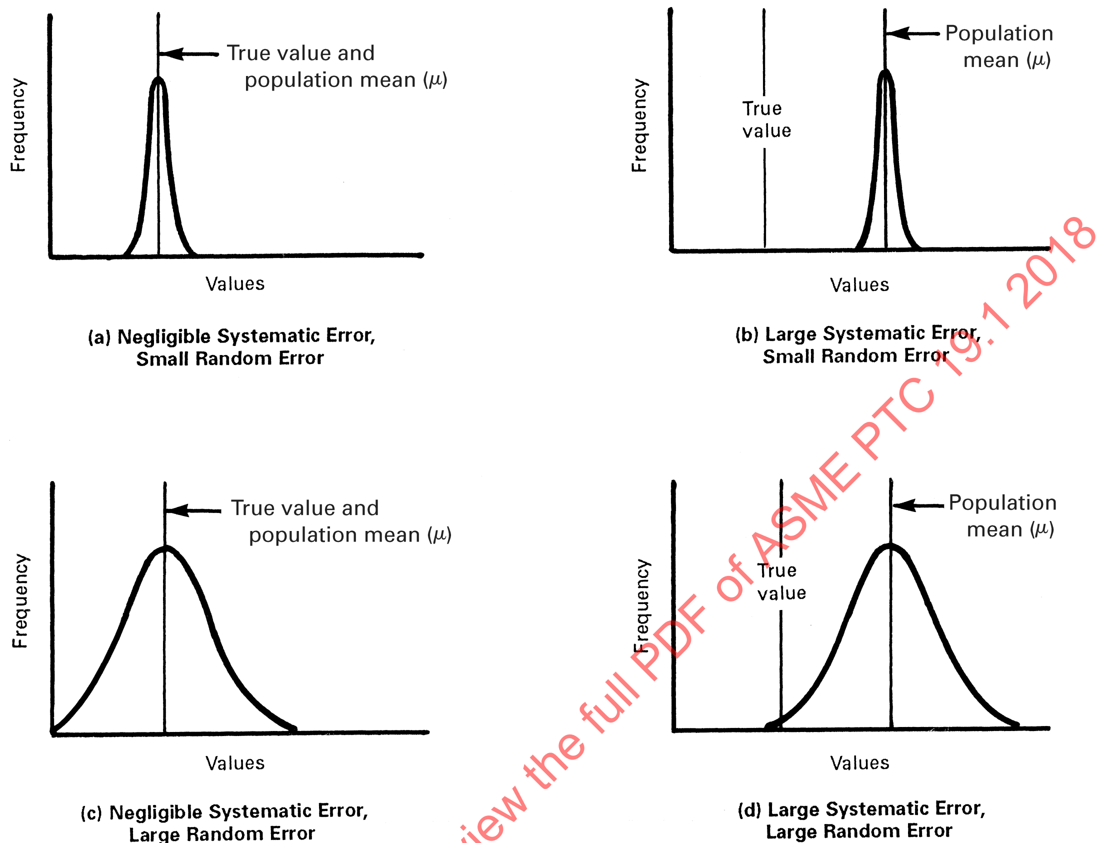
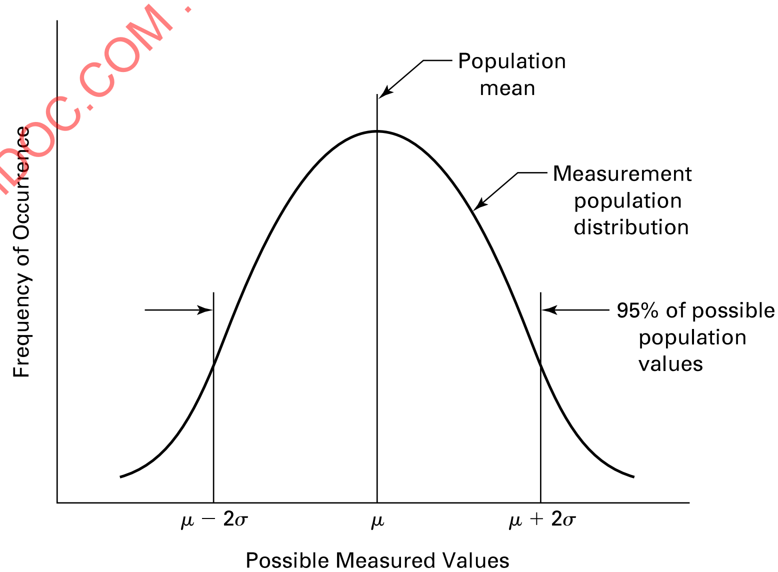


Figure 3-3.1-1 Population Distribution



3-3.2 Systematic Standard Uncertainty of a Measurand

Every measurement is influenced by multiple elemental systematic error sources, β_k. In successive measurements of a measurand at a set test condition, the values of these elemental systematic errors do not change. They are constant and therefore cannot be observed in collected test data. If an infinite number of measurements of a measurand were to be taken with the defined test process, the resulting population mean would still be in error due to the influence of these constant elemental systematic errors. For measurements with zero random error (see para. 3-2.1), every successive measurement will be exactly the same but all will be in error by the sum total of the elemental systematic errors that affect that measurement.

Each elemental systematic error contributes to every measurement affected by it in the same way at a set test condition. Since these errors are constant for the test, the error each imparts to an individual measurement is equivalent to the error imparted to the average value of successive measurements, $\overline{X}$ [as given by eq. (3-3-1)]. $\beta_{\overline{X}_k}$ represents each elemental systematic error affecting the average measurement where k denotes a specific elemental error source. While $\beta_{\overline{X}_k}$ is unknown, it can be postulated to come from a population of possible error values from which a single sample (error value) is drawn and imparted as an unknown and constant error to all the measurements and therefore to the average measurement of each elemental measurand at the test condition. Assuming or estimating the frequency distribution and standard deviation of this population of possible errors permits estimating the uncertainty of the test measurement average due to this single sample elemental systematic error. The elemental systematic standard uncertainty, $b_{\overline{X}_k}$, is defined as a constant value that estimates the dispersion of the population of possible $\beta_{\overline{X}_k}$ values at the standard deviation level.

All of the elemental systematic errors affecting a measurement combine to yield the total systematic error, $\beta_{\overline{X}}$, in the measurement's average. The total systematic standard uncertainty, $b_{\overline{X}}$, is defined as a constant value (at a set test condition) that estimates the dispersion of the population of possible $\beta_{\overline{X}}$ values at the standard deviation level.

Typically, total systematic standard uncertainty is quantified by

(a) identifying all significant elemental sources of systematic error for the measurement

(b) evaluating elemental systematic standard uncertainties as the standard deviations of the possible systematic error distributions

(c) for the TSM, combining the elemental systematic standard uncertainties into an estimate of the total systematic standard uncertainty for the measurement

In general, reducing the total systematic uncertainty is beneficial because

(a) it improves the sample mean as an estimator of the true value

(b) it reduces the risk of significant shifts in sample means from test to test if systematic errors change (e.g., when equipment is changed out for alternate equipment or is recalibrated)

3-3.2.1 Identifying Elemental Sources of Systematic Error. Attempting to identify all of the significant elemental sources of systematic error for a measurement is an important step in an uncertainty analysis. Failure to identify any significant source of systematic error will lead to an underestimate of measurand uncertainty. Attempting to identify all significant elemental sources of systematic error requires a thorough understanding of the test objectives and test process.

3-3.2.2 Evaluating Elemental Systematic Standard Uncertainties. Once all significant elemental sources of systematic error are identified, elemental systematic standard uncertainties for each source must be evaluated. Since the elemental systematic standard uncertainty is both constant and unknown at a given test condition, successive measurements do not provide data for direct computation of it using the standard deviation described in para 3-3.1. Therefore, the evaluation of an elemental systematic standard uncertainty requires that a standard deviation be estimated from published information, special data, or engineering judgment.

Note that the systematic and random errors in a calibration result are systematic as to their effect on test data. This is called "fossilization." This allows the calibration standard uncertainty to be one term in the combination of the test systematic standard uncertainties. (The random components in the calibration also become systematic terms in the test process, as these error sources do not add scatter to the test data as they did to the calibration data.)

3-3.2.2.1 Published Information. For some elemental systematic error sources, published information from calibration reports, instrument specifications, and other technical references may provide quantitative information regarding the dispersion of errors for an elemental systematic error source. The systematic uncertainty may be described in terms of a confidence interval, an ISO expanded uncertainty statement, or a multiple of a standard deviation.

If the published information is presented as a confidence interval (limits of error at a defined level of confidence), then the elemental systematic standard uncertainty is estimated as the confidence interval divided by a statistic that is appropriate for the frequency distribution of the error population. The specific value of this statistic must be selected on the basis of the defined confidence level and degrees of freedom associated with the confidence interval. For a normal distribution, the Student's t statistic is used. For a 95% confidence level and large degrees of freedom, the value of the Student's t statistic is approximated as 2, and eq. (3-3-4) would apply.

$$b_{\overline{X}_k} = \frac{B_{\overline{X}_k}}{2} \tag{3-3-4}$$

where

$B_{\overline{X}_k}$ = the published uncertainty assumed to be represented by normally distributed errors at a 95% confidence level

$b_{\overline{X}_k}$ = the systematic standard uncertainty in $\overline{X}$ for source k

Refer to Nonmandatory Appendix B for values of the Student's t statistic at other confidence levels and degrees of freedom. For situations in which the frequency distribution and degrees of freedom are unspecified, a uniform distribution and large degrees of freedom are often assumed. For situations involving other frequency distributions, refer to an appropriate statistics textbook.

If the published information is presented as an ISO expanded uncertainty at a defined coverage factor, then the elemental systematic standard uncertainty is estimated as the expanded uncertainty divided by the coverage factor.

If the published information is presented as a multiple of a standard deviation (e.g., "2-sigma" or "3-sigma"), then the elemental systematic standard uncertainty is estimated as the multiple of the standard deviation divided by the multiplier.

3-3.2.2.2 Special Test Data. Sometimes a separate, special test is needed to estimate the systematic error caused by a given source. An example is a nonuniform flow effect on the determination of the average velocity at a given location in a test article. In this case, the determination of the average velocity from a distribution of measurements at that location can be used to estimate the error when a smaller (probably more realistic) number of measurements is used to determine the average velocity at that location. Another example is the determination of an average temperature of a surface with a limited number of probes. If the surface temperature is almost uniform at the test location, then the average from the separate measurements will approximate the true average. If the temperature is not uniform due to heat conduction effects, then a two- or three-dimensional heat conduction analysis can be performed to estimate the error.

3-3.2.2.3 Engineering Judgment. It is often necessary to rely upon engineering judgment to quantify the dispersion of errors associated with an elemental error source. In these situations, it is customary to use engineering analyses and experience to estimate the limits of the elemental systematic error. A population of possible $\beta_{\overline{X}_k}$ values is chosen by the analyst. The standard deviation of this population is the estimate of systematic standard uncertainty for that error source.

In certain situations, knowledge of the physics of the measurement system will lead the analyst to believe that the limits of error are nonsymmetric (likely to be larger in either the positive or negative direction). For treatment of nonsymmetric systematic uncertainty, see subsection 7-2.

3-3.2.3 Combining Elemental Systematic Standard Uncertainties. Once evaluated, all of the elemental systematic standard uncertainties influencing a measurement are combined into an estimate of the total systematic standard uncertainty for the measurement $b_{\overline{X}}$. Provided all elemental systematic standard uncertainties are evaluated in terms of their influence on the measurand and in the units of the measurand, these elemental systematic standard uncertainties are combined per subsection 5-2 (using the TSM). Otherwise, these elemental systematic standard uncertainties are combined per subsection 6-4. In some cases, elemental systematic standard uncertainties may arise from the same elemental error source and are therefore correlated. See subsection 7-1 for a detailed discussion.

3-3.3 Combined Standard Uncertainty and Expanded Uncertainty

As previously discussed, the combined standard uncertainty in a measurement is the combination of uncertainty due to random error and uncertainty due to systematic error. The combined standard uncertainty of the measurement mean is calculated as follows (for TSM):

$$u_{\overline{X}} = \sqrt{(b_{\overline{X}})^2 + (s_{\overline{X}})^2} \tag{3-3-5}$$

where

$b_{\overline{X}}$ = the systematic standard uncertainty of the measurand

$s_{\overline{X}}$ = the random standard uncertainty of the measurand mean

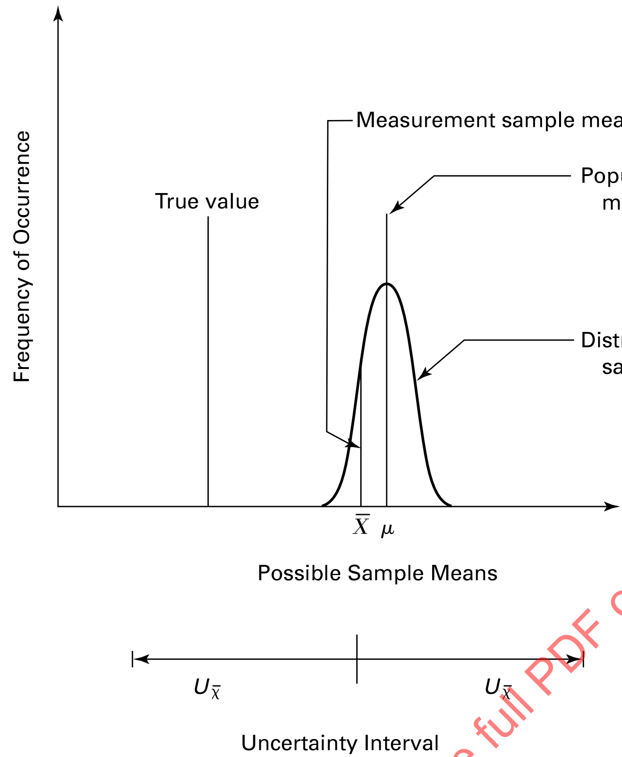
The expanded uncertainty of the measurement mean is the total uncertainty at a defined level of confidence. For applications in which a 95% confidence level is appropriate, the expanded uncertainty is calculated as follows (for TSM):

$$U_{\overline{X}} = 2u_{\overline{X}} \tag{3-3-6}$$

where the assumptions required for this simple equation are presented in subsection 5-4.

Expanded uncertainty is used to establish a confidence interval about the measurement mean that is expected to contain the true value. Thus, the interval $\overline{X} \pm U_{\overline{X}}$ is

Figure 3-3.3-1 Uncertainty Interval



expected to contain the true value with a 95% level of confidence, as seen in Figure 3-3.3-1.

Note that when using the MCM to determine an uncertainty, the random and systematic error distributions with the appropriate standard deviations (standard uncertainties) are used to determine the combined distribution for the measurement or result. The combined standard uncertainty is then the calculated standard deviation of that distribution and the expanded uncertainty at a given level of confidence determined from a coverage interval of the distribution (see para. 6-4.3).

3-4 PRETEST AND POST-TEST UNCERTAINTY ANALYSES

Although the analyst may be tempted to conduct an uncertainty analysis only once, there are benefits to conducting it both before and after the test, and then comparing the two results.

3-4.1 Pretest Uncertainty Analysis

The objective of a pretest analysis is to establish the expected uncertainty interval for a test result prior to the conduct of a test. A pretest uncertainty analysis is based on data and information that exist before the test, such as calibration histories, previous tests with similar instrumentation, prior measurement uncertainty analyses, expert opinions, and, if necessary, special tests.

A pretest uncertainty analysis should be considered because it allows preventive action to be taken prior to expending resources to conduct a test. The benefits of this proactive effort would be to make modifications to the test process to decrease the expected uncertainty to a level consistent with the overall test objectives, or to reduce the cost of the test while still acceptably attaining the objectives. Possible preventive actions include

(a) selecting alternative testing methods that rely upon different analysis procedures, testing under different conditions, and/or measurement of different measurands

(b) selecting alternative measurement methods by changing test instrumentation (type and/or quantity), calibration techniques, installation methods, and/or measurement locations

(c) changing sample sizes by changing sampling frequencies, changing test duration, and/or changing the number of repeat tests

(d) adjusting or substantiating test requirements

(e) reevaluating the test objectives

Additionally, a pretest uncertainty analysis facilitates communication between all parties to the test about the expected quality of the test. This can be essential to establishing agreement on any deviations from applicable test code requirements and can help reduce the risk that disagreements will surface after conducting the test.

3-4.2 Post-test Uncertainty Analysis

The objective of a post-test analysis is to establish the uncertainty interval for a test result after conducting a test. In addition to the data and information used to conduct the pretest uncertainty analysis, a post-test uncertainty analysis is based upon the additional data and information gathered for the test, including all test measurements, pretest and post-test instrument calibration data, etc. A post-test uncertainty analysis is recommended as it serves to

(a) validate the quality of the test result by demonstrating compliance with test requirements

(b) facilitate communication of the quality of the test result to all parties to the test

(c) facilitate interpretation of the quality of the test by those using the test result

Additionally, a post-test uncertainty analysis serves to validate the goodness of the pretest uncertainty analysis. If the post-test uncertainty results are much larger than the pretest uncertainty results, this may highlight an oversight of importance or a problem with one or more instruments or test processes. If the post-test uncertainty results are much smaller than the pretest uncertainty results, this may highlight far too much conservatism in one or more elements of the pretest analysis information. At a minimum, disparities of significance deserve some discussion. There may be sound technical and/or business reasons to further evaluate the differences and then possibly take corrective action for future tests.

Section 4
Defining the Measurement Process

4-1 OVERVIEW

The first step in a test uncertainty analysis is to clearly define the desired result and acceptable level of uncertainty for the result. Typically, the result is determined from multiple measured variables using a data reduction equation (DRE). Consideration must be given to the selection of the appropriate "true value" of each measured variable and the time interval for classifying errors as systematic or random. This Section provides an overview of how the measurement process should be defined.

4-2 SELECTION OF THE APPROPRIATE "TRUE VALUE"

Depending on the user's perspective, several measurement objectives or goals and hence corresponding "true values" (measurements with ideal zero error) of a measurand may exist simultaneously in a measurement process. For example, when analyzing a thermocouple measurement in a gas stream, several starting points or "true values" can be selected. The starting point for the analysis could be the "true value" defined as the metal temperature of the thermocouple junction, the gas stagnation temperature or junction temperature corrected for probe effects, or the mass flow-weighted average of the gas temperature at the plane of the instrumentation. Any of the above "true values" may be appropriate. The selection of the "true value" for the uncertainty analysis must be consistent with the goal of the measurement [4].

4-3 IDENTIFICATION OF ERROR SOURCES

Once the true value for a measurand has been defined, the errors associated with estimating the true value shall be identified. Examples of error sources include imperfect calibration corrections, uncontrolled test conditions, measurement methods, environmental conditions, and data reduction techniques. Estimates to quantify the limits of these errors are represented as uncertainties. These uncertainties in the measurement process can be grouped by source.

(a) calibration uncertainty

(b) uncertainty due to test article and/or instrumentation installation

(c) data acquisition uncertainty

(d) data reduction uncertainty

(e) uncertainty due to methods and other effects

4-3.1 Calibration Uncertainty

Each measurement instrument may introduce random and systematic uncertainties. The main purpose of the calibration process is to eliminate large, known systematic errors and thus reduce the measurement uncertainty to some "acceptable" level. Having decided on the "acceptable" level, the calibration process achieves that goal by exchanging the large systematic errors of an uncalibrated or poorly calibrated instrument for the smaller combination of systematic errors of the standard instrument and the random errors of the calibration. Calibrations are also used to provide traceability to known reference standards or physical constants, or both.

Requirements of military and commercial contracts have led to the establishment of extensive hierarchies of standards laboratories. In some countries, a national standards laboratory is at the apex of these hierarchies, providing the ultimate reference for every standards laboratory. As shown in Figure 4-3.1-1, each additional level in the calibration hierarchy adds uncertainty in the measurement process.

4-3.2 Uncertainty Due to Test Article and/or Instrumentation Installation

Test uncertainty can also arise from interactions between either the test instrumentation and the test media or the test article and the test facility.

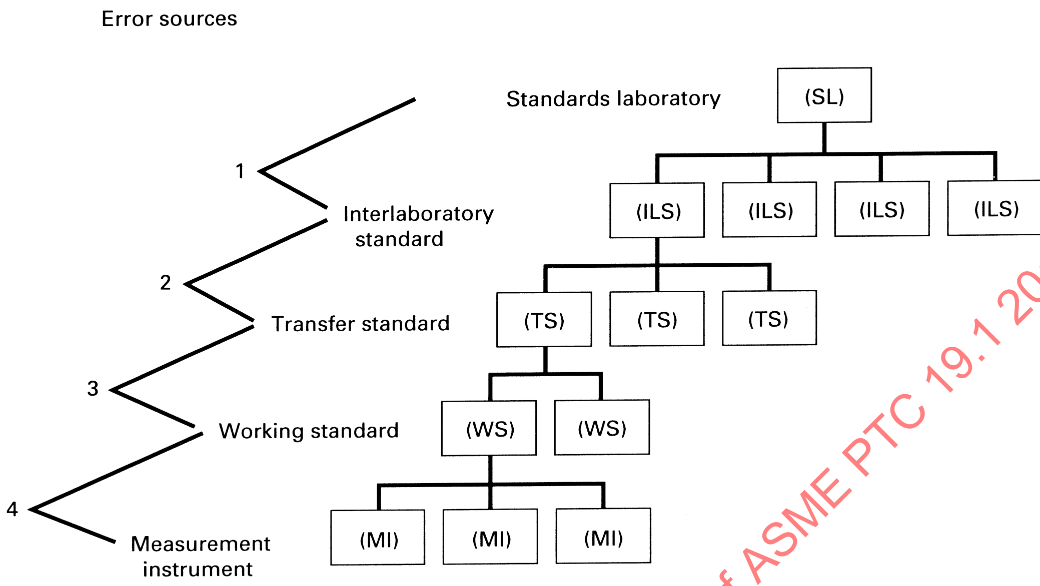
(a) Interactions between the test instrumentation and the test media

(1) Installation of sensors in the test media may cause intrusive disturbance effects. An example is the measurement of airflow in an air conditioning duct. Depending on the design, the pitot static probe may affect the measured total and static pressure and thus the calculated airflow.

(2) Environmental effects on sensors/instrumentation may exist when the sensors experience environmental effects different from those observed during calibration. These may include conduction, convection, and radiation on a sensor when installed in a gas turbine.

(b) Interactions between the test article and the test facility

Figure 4-3.1-1 Generic Measurement Calibration Hierarchy



(1) Test facility limitations can affect uncertainty. An example is an air conditioner bench tested in a laboratory but used in an automotive mechanics shop. The effect of the oily air can influence the quoted rating of the unit. A second example is the testing of a gas turbine aircraft engine in an altitude facility. The facility simulates altitude by lowering the ambient pressure at the test article exhaust and simulates forward speed by raising the inlet pressure at the engine inlet above ambient pressure. The facility inlet duct is of necessity significantly longer than the normal aircraft flight intake and so its boundary layer characteristics are significantly different. The engine performance test results must be corrected to account for this difference between the inlet duct in the facility and the intake on the aircraft.

(2) Facility limitations for testing may require extrapolations to other conditions. An example is the testing of an automotive engine. Although the fuel consumption of an automotive engine changes with altitude and speed, an automotive test facility may only be able to test at specified altitudes and speeds. Effects at other altitude conditions may need to be extrapolated.

4-3.3 Data Acquisition Uncertainty

Uncertainty in data acquisition systems can arise from errors in the signal conditioning, the sensors, the recording devices, etc. The best approach to minimizing the effects of many of these error sources is to perform overall system calibrations. By comparing known input values with their measured results, estimates of the data acquisition system uncertainty can be obtained. However, it is not always possible to do this. In these cases, it is necessary to evaluate each of the elemental uncertainties and to combine them to predict the overall uncertainty.

4-3.4 Data Reduction Uncertainty

Computations on raw data are often done to produce output (data) in a format more easily used in results calculations or application of calibration corrections. Typical error sources in this category stem from curve fits and computational resolution. With the recent advances in computer systems, the computational resolution error sources are often negligible; however, curve fit error can be significant. Other examples of data reduction uncertainty include

(a) the assumptions or constants contained in the calculation routines

(b) using approximating engineering relationships or violating their assumptions

(c) using an empirically-derived correlation such as empirical fluid properties

These additional uncertainties may be of either a systematic or a random nature, depending on their effect on the measurement.

4-3.5 Uncertainty Due to Methods and Other Effects

Uncertainties due to methods are defined as those additional uncertainty sources that originate from the techniques or methods inherent in the measurement process. These uncertainty sources—beyond those contained in calibration, installation sources, data acquisition, and data reduction—may significantly affect the uncertainty of the final results. An example is the determination of an average value of a variable in an environment characterized by nonuniform conditions.

Measurement requirements for a performance test often demand an average measurement of individual parameters. Most instrumentation, however, yields a point measurement of a parameter rather than an average measurement. While this point characteristic may be useful for other purposes, it raises a problem in determining performance level. In many instances, the quantity measured varies in space, making the point measurement inadequate. Thus, it often is necessary to install several measurement sensors at different spatial locations to account for spatial variations of the parameter being measured. Spatial variation effects are considered errors of method.

If an area-averaged value is desired, such as the average fluid velocity in a pipe at a cross section, then the definition of the average velocity is given by

$$\overline{V} = \frac{1}{A}\iint V(x, y)dA$$

and this is actually approximated using N measured velocity values V_i as

$$\overline{V} \approx \frac{1}{A}\sum_{i=1}^{N} V_i(\Delta A)_i$$

The average velocity is first determined using a reasonably large number of measurement locations for N, and then for test operation conditions a smaller number of locations is used. The error incurred due to method is the difference in the values of average velocity given by the two determinations.

4-4 CATEGORIZATION OF UNCERTAINTIES

This Standard delineates uncertainties by the effect of the error (i.e., systematic and random). This categorization approach supports the identification, understanding, and management of test uncertainties. If the nature of an elemental error is fixed over the duration of the defined measurement process, then the error contributes to the systematic uncertainty. If the error source tends to cause scatter in repeated observations of the defined measurement process, then the source contributes to the random uncertainty.

Because measurement uncertainties are categorized by the effect of the error, the time interval and duration of the measurement process can be important considerations and so must be clearly stated. The significance of this is discussed in para. 4-4.2. In addition, the objective of the test may affect the categorization, as discussed in para. 4-4.3.

4-4.1 Alternate Categorization Approach

An alternate approach, which is used in the ISO GUM, categorizes the uncertainties based on the method used to estimate them. Those evaluated with statistical methods are classified as Type A, while those evaluated by other means are classified as Type B. Depending on the selection of the defined measurement process, there may be no simple correspondence between random or systematic and Type A or Type B.

4-4.2 Time Interval Effects

Errors that may be fixed over a short time period may be variable over a longer time period. For example, calibration corrections, which are assumed fixed over the life of the calibration interval, can be considered variable if the process consists of a time interval encompassing several different calibrations. The time interval must be clearly specified to classify an error, and it may not always be the same interval as the test duration. For example, when comparing results among various laboratories, it may be more appropriate to classify an error as random rather than systematic even though that error may have been constant for the duration of any single test.

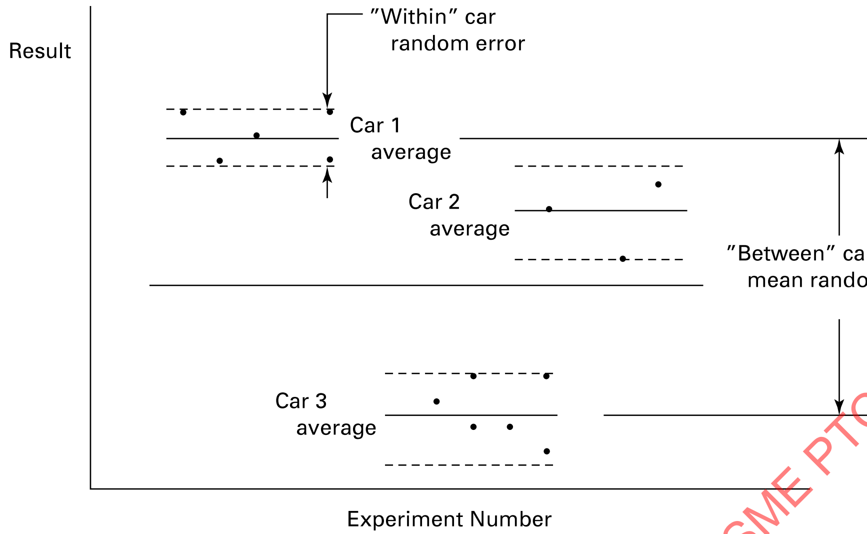
The effects of a time interval may also be important when considering the stability and control of a test process. The stability of a measurement method is a generic concept related to the closeness of agreement between test results. Process stability is estimated from observations of scatter within a data set and is treated as a random error. Variability in independent test results obtained under different test conditions, varying experimental setups, or configuration changes allow for additional between-test random errors.

4-4.3 Test Objective

The classification and number of error sources are often affected by the test objective. For example, if the test objective is to measure the average gas mileage of model "XYZ" cars, the variability among or between cars of the same model must be considered.

Random error obtained in a test from a given car would not include car-to-car variations and thus would not represent all random error sources. To observe the effect of the random error associated with car-to-car variability, the experiment would need to be run again using a random selection of different cars within the same model (see Figure 4-4.3-1). The total variation in the test result is greater than that observed from a test of a single given car. This variation would be more representative of the total random error associated with determining gas mileage for the fleet of model "XYZ" cars. Of course, if the data of interest is gas mileage of a given single car, then the estimated variation with testing the representative given car is an appropriate estimate for the random error. The same short-term and

Figure 4-4.3-1 "Within" and "Between" Sources of Data Scatter



long-term effects must be applied for other variables affecting gas mileage (temperature, altitude, humidity, road conditions, driver variations, etc.).

4-5 COMPARATIVE TESTING

The objective of a comparative test (also known as a back-to-back test) is to determine, with the smallest test uncertainty possible, the net effect of a design and/or performance change. The first test is run with the standard or baseline facility configuration. The second test is then run in the same facility with the design and/or performance change and, ideally, with instruments, setups, and calibrations identical to those used in the first test. The difference between the results of these tests is an indication of the effect of the design and/or performance change. Depending on whether common instrumentation, setups, and calibrations are used between comparative tests, the effects of correlated errors (see subsection 7-1) may cause the resulting test uncertainty of the difference between the test results to be less than the uncertainty of each separate test result. An example of back-to-back uncertainty analysis is shown in the example in para. 7-1.1.

A controlled back-to-back test is the ideal case of a comparative test wherein the same instrumentation is used for both tests. This assumes that there is no change in instrument performance due to damage or other factors between the pre-change test and post-change tests. This also assumes that the tests are performed within a reasonable time of each other such that the initial instrument calibrations are not voided. In this case, all instruments are perfectly correlated and systematic uncertainty for the combined result is significantly reduced. The random uncertainties of the two test results then will be important for determining the uncertainty in the comparison.

Another common form of the comparative test, known as an uncontrolled back-to-back test, uses the same test methods for both the pre-change test and post-change test but does not use all of the same instruments between tests. Use of different instruments could be the result of multiple causes, including damage to instruments between tests, lapse of instrument calibration during the time between the pre-change and post-change tests, or engineering judgment. In this case, replacement instruments chosen should be similar in accuracy and specifications to the original instruments so that the effects of correlated errors (see subsection 7-1) may reduce the systematic uncertainty of the difference between the test results.

Section 5
Uncertainty of a Measurement

5-1 RANDOM STANDARD UNCERTAINTY OF THE MEAN

5-1.1 General Case

For $\overline{X}$ determined as the average of N measurements, the appropriate random standard uncertainty of the mean, $s_{\overline{X}}$, is given by eq. (3-3-3). This type of estimate is an ISO Type A estimate.

In a sample of measurements, the degrees of freedom is the sample size, N. When a statistic is calculated from the sample, the degrees of freedom associated with the statistic is reduced by one for every estimated parameter used in calculating the statistic. For example, from a sample of size N, $\overline{X}$ is calculated by eq. (3-3-1). The sample standard deviation, s_X, and the random standard uncertainty of the mean, $s_{\overline{X}}$, are calculated from eqs. (3-3-2) and (3-3-3), respectively, and each has N − 1 degrees of freedom, υ:

$$\nu = N - 1 \tag{5-1-1}$$

because $\overline{X}$ (based on the same sample of data) is used in the calculations of both quantities.

5-1.2 Using Previous Values of $s_{\overline{X}}$

In some test situations, the measurement of a variable may be only a single measurement or an average of measurements taken over a short time frame, as with a computer-based data acquisition system. In this latter case, the time frame over which the measurements are taken may be on the order of milliseconds or less, while the random variations in the process may be on the order of seconds or minutes or even days. This "short time frame averaged" value should then be handled in the same manner as a single measurement.

The random standard uncertainty for a single measurement must be estimated from historical or previous data taken over similar test conditions. This protocol is typically followed when performing a pretest measurement-uncertainty estimate.

Estimating the random standard uncertainty of a single measurement must be done by evaluating previous measurements of the parameter taken over similar test conditions. For example, taking multiple measurements as a function of time while holding all other conditions constant would identify the random variation associated with the measurement system and the unsteadiness of the test condition. If the sample standard deviation of the variable being measured is also expected to be representative of other possible random variations in the measurement (e.g., repeatability of test conditions, variation in test configuration, etc.), then these additional error sources will have to be varied while the multiple data measurements are taken to determine the standard deviation.

Another situation where previous values of a variable would be useful is when a small sample size, N, is used to calculate the mean value, $\overline{X}$, of a measurement. If a much larger set of previous measurements of the same test conditions is available, then it could be used to calculate a more appropriate standard deviation for the current measurement [5]. Typically, these larger data sets are taken in the early phases of an experiment program. Once the random variation of the test variables is understood, then this information can be used to streamline the test procedures by reducing the number or measurements taken in the later phases of the test.

When N_p previous values, X_{P_j}, are known for the quantity being measured, the sample standard deviation for the variable can be calculated as

$$s_X = s_{X_p} = \left[\frac{1}{N_{P-1}} \sum_{j=1}^{N_p} \left(X_{P_j} - \overline{X}_P \right)^2 \right]^{1/2} \tag{5-1-2}$$

where

$$\overline{X}_p = \frac{1}{N_p} \sum_{j=1}^{N_p} X_{P_j} \tag{5-1-3}$$

The appropriate random standard uncertainty of the mean for the current measurement, $\overline{X}$, is then

$$s_{\overline{X}} = \frac{s_X}{\sqrt{N}} \tag{5-1-4}$$

where

N = the number of current measurements averaged to determine $\overline{X}$

The number of degrees of freedom for this random standard uncertainty of the mean, $s_{\overline{X}}$, is

$$\upsilon = N_p - 1 \tag{5-1-5}$$

This estimate of the random standard uncertainty is an ISO Type A estimate since it is obtained from data. The case where the current data sample is only a single measurement is handled with $N = 1$ in eq. (5-1-4).

5-1.3 Using Elemental Random Error Sources

Another method of estimating the random standard uncertainty of the mean for a measurement is from information about the elemental random error sources in the entire measurement process. If all the random standard uncertainties are expressed in terms of their contributions to the measurement, then the random standard uncertainty for the measurement mean is the root-sum-square of the elemental random standard uncertainties of the mean from all sources divided by the square root of the number of current readings, N, averaged to determine $\overline{X}$:

$$s_{\overline{X}} = \frac{1}{\sqrt{N}}\left[\sum_{k=1}^{K}(s_{\overline{X}_k})^2\right]^{1/2} \tag{5-1-6}$$

where

K = the total number of random error (or uncertainty) sources.

Each of the elemental random standard uncertainties of the mean, $s_{\overline{X}_k}$, is calculated using the methods described in para. 5-1.1 or para. 5-1.2, depending on which is appropriate. If in each of the N measurements of the variable X, the output of an elemental component is averaged N_k times to obtain $s_{\overline{X}_k}$, then the method in para. 5-1.2 would apply.

The degrees of freedom for the estimated random standard uncertainty of the mean, $s_{\overline{X}}$, is dependent on the information used to determine each of the elemental random standard uncertainties of the mean and is calculated as

$$\upsilon = \frac{\left(\sum_{k=1}^{K}(s_{\overline{X}_k})^2\right)^2}{\sum_{k=1}^{K}\frac{(s_{\overline{X}_k})^4}{\upsilon_k}} \tag{5-1-7}$$

where

υ_k = the appropriate degrees of freedom for $s_{\overline{X}_k}$ and is obtained from eq. (5-1-1) or eq. (5-1-5), as appropriate

When all error sources have large sample sizes, the calculation of υ is unnecessary. However, for small samples, when combining elemental random standard uncertainties of the mean by the root-sum-square method [eq. (5-1-6)], the degrees of freedom, υ, associated with the combined random standard uncertainty is calculated using the Welch-Satterthwaite formula [6], eq. (5-1-7).

5-1.4 Using Estimates of Sample Standard Deviation

In a pretest uncertainty analysis, previous information might not be available to estimate the sample standard deviation as discussed in para 5-1.2 or para. 5-1.3. In this case, an estimate of the sample standard deviation, s_X, would be made using engineering judgment and the best available information. This type of uncertainty estimate would be an ISO Type B estimate.

5-2 SYSTEMATIC STANDARD UNCERTAINTY OF A MEASUREMENT

The systematic standard uncertainty, $b_{\overline{X}}$, of a measurement was defined in para. 3-3.2 as a value that quantifies the dispersion of the systematic error associated with the mean. The true systematic error, β, is the unknown, but $b_{\overline{X}}$ is the evaluated so that it represents an estimate of the standard deviation of the distribution for the possible β values. It should be noted that while $b_{\overline{X}}$ is an estimate of the dispersion of the systematic errors in a measurement, the systematic error that is present in specific measurement is a fixed single value of β.

The systematic standard uncertainty of the measurement is the root-sum-square of the elemental systematic standard uncertainties, $b_{\overline{X}_k}$, for all sources (TSM).

$$b_{\overline{X}} = \left[\sum_{k=1}^{K}\left(b_{\overline{X}_k}\right)^2\right]^{1/2} \tag{5-2-1}$$

where

$b_{\overline{X}_k}$ = each estimate of the standard deviation of the k^{th} elemental error source

K = the total number of systematic error sources

Note that in eq. (5-2-1), all of the elemental systematic standard uncertainties are expressed in terms of their contributions to the measurement.

For each systematic error source in the measurement, the elemental systematic standard uncertainty must be estimated from the best available information. Usually these estimates are made using engineering judgment (and are therefore ISO Type B estimates). Sometimes previous data are available to make estimates of uncertainties that remain fixed during a test (and are therefore ISO Type A estimates). If any of the elemental systematic uncertainties are nonsymmetrical, then the method given in para. 7-2.1 should be used to determine the systematic standard uncertainty of the measurement.

There can be many sources of systematic error in measurement, such as the calibration process, instrument systematic errors, transducer errors, and fixed errors of method. Also, environmental effects, such as radiation effects in a temperature measurement, can cause systematic errors of method. There usually will be some elemental systematic standard uncertainties that will be dominant. Because of the resulting effect of combining the elemental uncertainties in a root-sum-square manner, the larger or dominant ones will control the systematic uncertainty in the measurement; however, one should be very careful to identify all significant sources of fixed error in the measurement.

5-3 CLASSIFICATION OF UNCERTAINTY SOURCES

As discussed in para. 1-2.2.3, the ISO Guide classifies uncertainties by source, as either Type A or Type B [2]. Type A uncertainties are the calculated standard deviations obtained from data sets. Type B uncertainties are those that are estimated or approximated rather than calculated from data. Type B uncertainties are also given as estimated standard deviations.

In this Code, uncertainties are classified by their effect on the measurement, either random or systematic, rather than by their source. This effect classification was chosen since most test operators are concerned with how errors in the test will affect the measurements.

There may be situations when it is convenient to classify elemental uncertainties by both effect and source. Such classifications may be useful in international test programs. This Code recommends the following nomenclature for dual classifications:

$b_{\overline{X}_k,A}$ = elemental systematic standard uncertainty calculated from data, as in a calibration process

$b_{\overline{X}_k,B}$ = elemental systematic standard uncertainty estimated from the best available information

$s_{\overline{X}_k,A}$ = elemental random standard uncertainty calculated from data

$s_{\overline{X}_k,B}$ = elemental random standard uncertainty estimated from best available information

5-4 COMBINED STANDARD AND EXPANDED UNCERTAINTY OF A MEASUREMENT

For simplicity of presentation, a single value is often preferred to express the estimate of the error between the mean value, $\overline{X}$, and the true value with a defined level of confidence. The interval

$$\overline{X} \pm U_{\overline{X}} \tag{5-4-1}$$

represents a band about $\overline{X}$ within which the true value is expected to lie with a given level of confidence (see Figure 3-3.3-1). The uncertainty interval is composed of both the systematic and random uncertainty components.

The general form of the expression for determining the uncertainty of a measurement is the root-sum-square of the systematic and random standard uncertainties for the measurement, with this quantity defined as the combined standard uncertainty, $u_{\overline{X}}$, (TSM) [2]:

$$u_{\overline{X}} = \sqrt{(b_{\overline{X}})^2 + (s_{\overline{X}})^2} \tag{5-4-2}$$

where

$b_{\overline{X}}$ = the systematic standard uncertainty [eq. (5-2-1)]

$s_{\overline{X}}$ = the random standard uncertainty of the mean [eq. (3-3-3), eq. (5-1-4), or eq. (5-1-6) as appropriate]

In order to express the uncertainty at a specified confidence level using the TSM, the combined standard uncertainty must be multiplied by an expansion factor taken as the appropriate Student's t value for the required confidence level (see Nonmandatory Appendix B). Depending on the application, various confidence levels may be appropriate. The Student's t is chosen on the basis of the level of confidence desired and the degrees of freedom. The degrees of freedom used is a combined degrees of freedom based on the separate degrees of freedom for the random standard uncertainty and the elemental systematic standard uncertainty (see Nonmandatory Appendix B). A t value of 1.96 (usually taken as 2) corresponds to large degrees of freedom and defines an interval with a level of confidence of approximately 95%. This expansion factor of 2 is used for most engineering applications. For other confidence levels or fewer degrees of freedom, see Nonmandatory Appendix B.

The expanded uncertainty for a 95% level of confidence and large degrees of freedom (t = 2) is calculated per the TSM:

$$U_{\overline{X}} = 2u_{\overline{X}} \tag{5-4-3}$$

where

$u_{\overline{X}}$ = the combined standard uncertainty [eq. (5-4-2)]

The expression for the expanded uncertainty given in eq. (5-4-3) applies only when the measurement $\overline{X}$ is the desired result of the experiment. If several variables are measured and used in a DRE, then the techniques in Section 6 are used.

For the MCM propagation of uncertainties, the coverage interval is determined as described in para. 6-4.3.

5-4.1 Example

A digital thermometer is used to measure the average temperature of a circulating water bath being used in an experiment. The experiment lasts a total of 30 min. Temperature measurements are collected every minute, resulting in a total of 31 data points, as presented in Table 5-4.1-1.

Table 5-4.1-1 Circulating Water-Bath Temperature Measurements (Example 5-4.1)

Elapsed Time, min	Measured Temperature, °C	Elapsed Time, min	Measured Temperature, °C	Elapsed Time, Min	Measured Temperature, °C
0	85.11	11	85.28	21	85.23
1	84.89	12	85.11	22	85.12
2	85.07	13	84.80	23	85.43
3	84.77	14	84.79	24	84.50
4	85.24	15	85.22	25	85.22
5	84.72	16	85.05	26	85.39
6	85.00	17	84.58	27	84.74
7	85.39	18	85.20	28	85.35
8	84.72	19	85.14	29	84.75
9	85.50	20	85.05	30	84.56
10	85.18				

(a) Uncertainty Due to Random Error. The uncertainty due to the random error of the average temperature measurement is evaluated using the steps herein.

(1) The sample mean, or average value, of the temperature measurements is determined using eq. (3-3-1).

$$\overline{X} = \frac{1}{N}\sum_{j=1}^{N} X_j = 85.04$$

(2) The sample standard deviation is determined using eq. (3-3-2).

$$s_X = \sqrt{\sum_{j=1}^{N} \frac{\left(X_j - \overline{X}\right)^2}{N-1}} = 0.28°\text{C}$$

(3) The random standard uncertainty of the sample mean is determined using eq. (3-3-3).

$$s_{\overline{X}} = \frac{s_X}{\sqrt{N}} = 0.05°\text{C}$$

(b) Uncertainty Due to Systematic Error. The uncertainty due to the systematic error of the average circulating water-bath temperature measurement is evaluated by the steps herein.

(1) Identify all significant elemental sources of systematic error for the measurement.

(2) Evaluate elemental systematic standard uncertainties as the standard deviations of the possible systematic standard error distributions.

(3) Combine the elemental systematic standard uncertainties into an estimate of the total systematic standard uncertainty for the measurement.

For the purpose of this example, a summary of this evaluation is presented in Table 5-4.1-2. Refer to para. 3-3.2 and subsection 5-2 for further discussion of the process for identifying, evaluating, and combining elemental systematic uncertainties.

The systematic standard uncertainty of the temperature is calculated using eq. (5-2-1).

$$b_{\overline{X}} = \left[\sum_{i=1}^{k} \left(b_i^2\right)\right]^{1/2} = 0.07°\text{C}$$

Note that because of the root-sum-square combination of the elemental sources, only two of the five uncertainties contribute meaning.

(c) Expanded Uncertainty (TSM). The expanded uncertainty (TSM) of the average circulating water-bath temperature measurement is evaluated using eqs. (5-4-2) and (3-3-6).

$$U_{\overline{X}} = 2\sqrt{\left(b_{\overline{X}}\right)^2 + \left(s_{\overline{X}}\right)^2} = 0.22°\text{C}$$

Therefore, the true average temperature of the circulating bath during the experiment is expected to lie within the following interval with 95% level of confidence:

$$\overline{X} \pm U_{\overline{X}} = 85.4 \pm 0.22°\text{C}$$

Table 5-4.1-2 Systematic Standard Uncertainty of Average Circulating Water-Bath Temperature Measurement (Example 5-4.1)

Description of Systematic Uncertainty Source	Elemental Systematic Standard Uncertainty, °C	ISO Types
Calibration of digital thermometer	0.05	B
Environmental influences (ambient temperature, humidity, etc.) on digital thermometer	0.005	A
Effects of conduction heat transfer surroundings	0.0005	B
Uniformity of circulating water bath (spatial uncertainty)	0.05	A
Effects of radiation heat transfer	Negligible	B

Section 6
Uncertainty of a Result Calculated From Multiple Parameters

6-1 RESULTS CALCULATED FROM MULTIPLE PARAMETERS

Calculated results, such as the determination of efficiency, are not usually measured directly. Instead, more basic parameters, such as temperature and pressure, are either measured or assigned values (such as properties from tabulated values) and the required result is calculated as a function of these parameters. The random and systematic uncertainties in these measurements or assigned values of the parameters propagate through the functional relationship between the results and the parameters.

$$R = f(X_1, X_2, \ldots, X_i) \tag{6-1-1}$$

This produces a random, s_R, and systematic, b_R, standard uncertainty in the calculated result.

In this Section, methods of calculating these uncertainties in the result will be discussed. Two methods for propagating the uncertainties are presented, the Taylor series method (TSM) and the Monte Carlo method (MCM).

There are two approaches to calculating s_R, a direct (multiple results) method and a propagation method. The choice of approach depends on whether, at a given test condition, multiple results are available or only a single result is available. There is no equivalent to the direct method for calculating b_R, in the result, so a propagation approach is always used.

6-1.1 Single and Repeated Tests

In some experimental situations, a set of parameters, $\overline{X}_i$, is measured and a single result, R, is calculated for some given test condition. Examples are sample-to-sample type tests in which the test sample is destroyed, e.g., determining the ultimate strength or the heating value of material. In such cases, some of the parameters may be based on single measurements and others may be the mean values based on N_i repetitions. N_i can be different for each X_i. The result, R, is expressed in terms of the average or assigned values of the independent parameters, X_i, that enter into the result. That is,

$$R = f(\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_i) \tag{6-1-2}$$

where the subscript i signifies the total number of parameters involved in R, and the average values of the independent parameters are obtained as

$$\overline{X}_i = \frac{1}{N_i}\sum_{j=1}^{N_i} X_{i_j} \tag{6-1-3}$$

where

N_i = the number of measurements of X_i

In such cases, both s_R and b_R must be determined using a propagation method, since multiple results at a given test condition are not available for a direct calculation of s_R.

6-1.2 Multiple Results: Test With the Result Calculated Multiple Times at a Given Condition

If multiple results are calculated at a given test condition, then a sample distribution of results is obtained. This typically occurs in one of two ways. In the case of a sample-to-sample type of experiment, as in para. 6-1.1, repeated tests on multiple samples of the same material yield such a sample of results. The other common case is the time-wise type of experiment performed over a period of time, e.g., a steady-state test for turbine efficiency. At a given test condition, a set of parameters, X_i, is measured multiple times and multiple results, R_m, can be calculated from each set of measurements as

$$R_m = R[(X_1)_m, (X_2)_m, \ldots, (X_i)_m] \tag{6-1-4}$$

In both sample-to-sample and time-wise experiments, the average result is given by

$$\overline{R} = \frac{\sum_{m=1}^{M} R_m}{M} \tag{6-1-5}$$

where

M = the number of results at the given test condition

In such situations, values of s_R can be determined from both the direct method using an equation analogous to eq. (3-3-2) and one or both of the TSM and MCM propagation methods. The multiple values of s_R can be compared with one another. The implications of this comparison in identifying the presence of correlated behavior in the random errors affecting multiple parameters is discussed in para. 6-3.1.2.

As mentioned previously, there is no equivalent to the direct method for calculating systematic uncertainty in the result, so b_R must be determined from a propagation method.

6-1.3 Determining Uncertainties in Results Calculated From Multiple Parameters

The following subsections present details of the techniques used to determine the random and systematic uncertainties in results calculated from multiple parameters. In subsection 6-2, the direct method of calculating the random uncertainty of a set of multiple results at a given test condition is presented; this is applicable to the cases described in para. 6-1.2.

In subsection 6-3, the TSM of propagation to determine both the random and systematic uncertainties in a result is presented; it is applicable in both single-result and multiple-results cases. For the multiple-results cases discussed in para. 6-1.2, two estimates of s_R can thus be made (using the direct and Taylor series methods) and compared.

In subsection 6-4, the MCM of propagation to determine both the random and the systematic uncertainties in a result is presented; this also is applicable in both single-result and multiple-results cases. For the multiple-results cases discussed in para. 6-1.2, two estimates of s_R can thus be made (using the direct and Monte Carlo methods) and compared.

6-2 DIRECT METHOD OF DETERMINING RANDOM STANDARD UNCERTAINTY FROM A SAMPLE OF MULTIPLE RESULTS

6-2.1 Direct Calculation of the Random Standard Uncertainty From a Sample of Multiple Results

Following eq. (3-3-2), the estimate of the standard deviation of the distribution of M results at a given test condition is

$$s_R = \left(\frac{\sum_{m=1}^{M} (R_m - \overline{R})^2}{M - 1} \right)^{1/2} \qquad (6\text{-}2\text{-}1)$$

The random standard uncertainty of the mean result is estimated directly from the sample standard deviation and is given by

$$s_{\overline{R}} = \frac{s_R}{\sqrt{M}} \qquad (6\text{-}2\text{-}2)$$

6-2.2 Some Practical Consideration for Multiple Results at a Given Test Condition

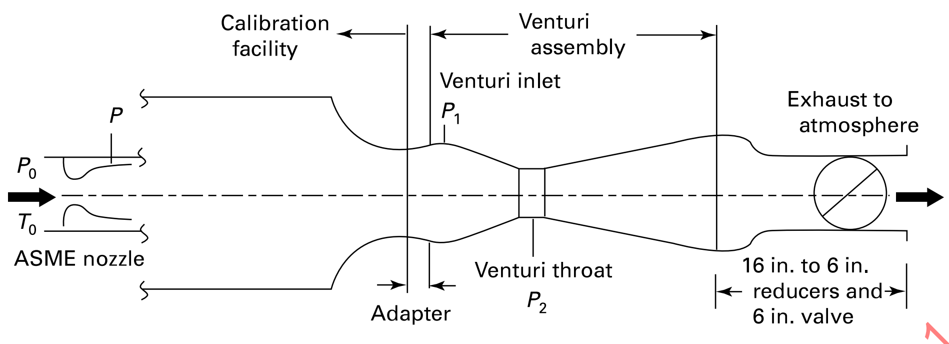
In time-wise type experiments, variations in a "steady state" condition may lead to correlated variations in different parameters. An example is a heat exchanger test to determine heat rate at a constant flow rate condition, where an uncontrolled upward drift with time of the inlet fluid temperature would likely result in an upward drift with time of the outlet fluid temperature. Such correlations caused by time-varying error sources that affect the separate parameter measurements in the same way are automatically taken into account in the direct method of calculating the random standard uncertainty of the result. This is not the case for calculations of the random standard uncertainty of the result using either the Taylor series or Monte Carlo propagation methods, which require inclusion of special terms to account for such correlations. In practice, the special correlation terms have rarely been included in the analysis. This is discussed in detail and with an example in subsection 6-3.

When tests are repeated under similar operating conditions, these generate multiple data sets for the measured parameters. The statistics found by combining these multiple data sets may be used to estimate the variations in the result that might be due to the control of test operating conditions, or use of different test rigs, instrumentation, or test location. Whereas these influences might normally be considered systematic errors during repeated tests, the duplicated tests can randomize these systematic errors, providing error estimates from the statistical variations in the combined data pool [7]. The overall reported result will usually be combined to provide the mean of the multiple results, $\overline{R}$.

Careful consideration should be given to designing the test series to average as many causes of variation as possible within cost constraints. The test design should be tailored to the specific situation. For example, if experience indicated that time-to-time and test apparatus-to-apparatus variations are significant, a test design that averages multiple test results on one rig or for only one day may produce optimistic random uncertainty estimates compared to testing several rigs, each monitored several times over a period of several days. The list of test variation causes are many and may include the above plus environmental and test crew variations. Historic data are invaluable for studying these effects. A statistical technique called analysis of variance (ANOVA) is useful for partitioning the total variance by source [8].

When more than one test is conducted with the same instrument package (i.e., repeated tests), the uncertainty of the average test result may be reduced from that for one test because of the reduction in the random uncertainty of the average. However, systematic uncertainty will remain the same as for a single test provided the measurement system and instrumentation do not change during the test,

Figure 6-3.1-1 Venturi Calibration



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and influences from environmental effects do not change between tests.

6-3 TAYLOR SERIES METHOD (TSM) OF PROPAGATION FOR DETERMINING RANDOM AND SYSTEMATIC UNCERTAINTIES OF A RESULT

6-3.1 Random Standard Uncertainty of a Result (TSM)

The random standard uncertainty of a single test result using the TSM is given by

$$s_R = \left[\sum_{i=1}^{l} \left(\theta_i s_{\overline{X}_i} \right)^2 + \left(\begin{array}{c} \text{random error} \\ \text{correlation term} \end{array} \right) \right]^{1/2} \quad (6\text{-}3\text{-}1)$$

where

$$\theta_i = \frac{\partial R}{\partial \overline{X}_i} \quad (6\text{-}3\text{-}2)$$

is the sensitivity coefficient for the result R given by eq. (6-1-1).

The relative random standard uncertainty is found by nondimensionalizing eq. (6-3.1) by dividing by the result so that the TSM gives

$$\frac{s_R}{R} = \left[\sum_{i=1}^{l} \left(\theta_i' \frac{s_{\overline{X}_i}}{\overline{X}_i} \right)^2 + \left(\frac{1}{R} \right)^2 \left(\begin{array}{c} \text{random error} \\ \text{correlation terms} \end{array} \right) \right]^{1/2} \quad (6\text{-}3\text{-}3)$$

where the relative (nondimensional) sensitivity coefficient (θ_i') is given by

$$\theta_i' = \frac{\dfrac{\partial R}{R}}{\dfrac{\partial \overline{X}_i}{\overline{X}_i}} = \frac{\overline{X}_i}{R} \left(\frac{\partial R}{\partial \overline{X}_i} \right) \quad (6\text{-}3\text{-}4)$$

6-3.1.1 Example: Random Uncertainty Determination in Venturi Discharge Coefficient Calibration. As reported by Hudson et. al [9] and shown in Figure 6-3.1-1, a venturi meter was calibrated in a commercial facility using an ASME nozzle as the flow standard.

The venturi discharge coefficient determined in this test is a function of the standard flow rate, W_{std}; the venturi inlet pressure, P_1, and temperature, T_1; the throat pressure, P_2; and the venturi inlet diameter, D_1, and throat diameter, D_2. It can be represented as

$$C_d = f(W_{std}, P_1, P_2, T_1, D_1, D_2)$$

Values of the discharge coefficient, C_d, were determined in a sequence of 11 different tests at chosen Mach and Reynolds number conditions. The standard ASME nozzle was choked at all conditions. At a given test condition, ten data scans were taken and average values of the venturi pressures and temperatures calculated using eq. (6-1-3). These average values were used to calculate average Mach and Reynolds numbers, and used in eq. (6-1-2) to calculate a value for C_d.

For each test condition, the random standard uncertainty of the result, C_d, was determined using both the TSM and eq. (6-3-1) with the correlated error terms set to zero and the direct method and eqs. (6-1-4) and (6-2-1). The comparison of the application of the two methods is shown in Table 6-3.1-1.

The random uncertainties calculated by TSM propagation were from 2 to 51 times larger than those determined with the direct method. The propagation method treated the random uncertainties in the two venturi pressures as independent. Figure 6-3.1-2 is a plot of these two pressures normalized to the critical flow nozzle inlet total pressure for a particular test that shows the variations of the two pressures are not independent.

The same trend was seen for all the test conditions. The fact that the pressures varied was a function of the test facility control. The variations in the pressure measurements were not truly random; they were correlated. This

Table 6-3.1-1 Comparison of TSM and Direct Method Values of Random Standard Uncertainty in C_d

Test	Mach	Re × 10^{-6}	TSM/Direct
1	0.20	1.0	17
2	0.19	1.0	39
3	0.20	1.1	51
4	0.20	2.9	30
5	0.20	6.0	19
6	0.50	1.0	7
7	0.50	3.0	14
8	0.49	5.8	2
9	0.70	1.5	2
10	0.70	3.0	8
11	0.68	5.9	3

correlation was not considered in the propagation method but was taken into account automatically in the direct method (even though the test operators were previously totally unaware of the correlation).

6-3.1.2 Some Practical Considerations in Determining Random Standard Uncertainty of a Result. The direct method is always preferred because it takes into account any correlated random error effects in the sample of results, whether these effects are recognized or not. The only situation in which a propagation method estimate of random uncertainty should be the sole estimate made is one in which only a single test result is determined and the direct method cannot be applied. In all other situations, calculations of random uncertainty by a propagation method can be compared to that determined by the direct method, and the comparison used as an indicator of the presence or absence of correlated random error effects. Sometimes in time-wise tests the correlated random errors are not "random" but rather results from a drift with time of multiple variables. All time-wise tests should be designed so that multiple results can be calculated at each constant test condition if at all possible.

Although the effect of correlated random errors in the example in para. 6-3.1.1 caused the propagation technique estimate to be greater than that of the direct method, the opposite can also occur depending on the form of eq. (6-1-1). This was illustrated by results from a full-scale rocket engine ground test and also a laboratory-scale cold flow facility [10].

6-3.2 Systematic Standard Uncertainty of a Result (TSM)

The systematic standard uncertainty of a single test result using the TSM is given by

$$b_R = \left[\sum_{i=1}^{I} \left(\theta_i b_{\overline{X}_i}\right)^2 + \begin{pmatrix} \text{systematic} \\ \text{correlation terms} \end{pmatrix} \right]^{1/2} \tag{6-3-5}$$

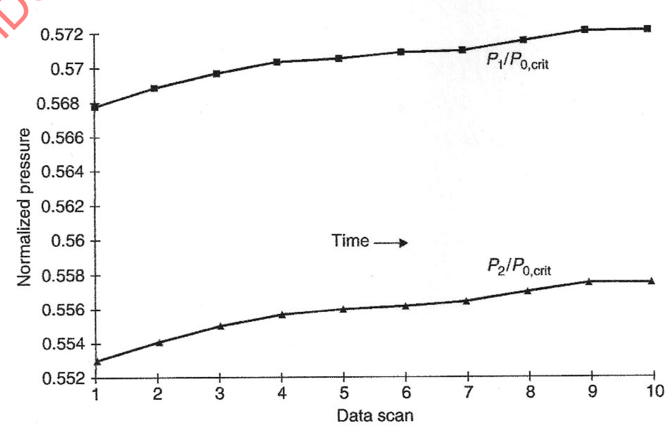
The relative systematic uncertainty of a result is

$$\frac{b_R}{R} = \left[\sum_{i=1}^{I} \left(\theta_i' \frac{b_{\overline{X}_i}}{\overline{X}_i}\right)^2 + \left(\frac{1}{R}\right)^2 \begin{pmatrix} \text{systematic} \\ \text{correlation terms} \end{pmatrix} \right]^{1/2} \tag{6-3-6}$$

The symbol $b_{\overline{X}_i}$ is the systematic standard uncertainty of the measured parameter (see subsection 5-2).

The correlation terms in eqs. (6-3-5) and (6-3-6) will be zero if all the systematic error sources for all the parameters used to determine R are totally independent. However, in many cases, multiple parameters share a

Figure 6-3.1-2 Normalized Venturi Inlet and Throat Pressures for a Test



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common systematic error source, such as a calibration standard error. An example would be two thermocouples calibrated against the same standard. In this case, the systematic error correlation terms would be nonzero. Their magnitude can be calculated using the methodology in para. 7-1.1.

6-3.3 Combined Standard Uncertainty and Expanded Uncertainty of a Result (TSM)

The general form of the expression for determining the combined standard uncertainty of a result is the root-sum-square of both the systematic and the random standard uncertainty of the result:

$$u_R = \left[(b_R)^2 + (s_R)^2 \right]^{1/2} \tag{6-3-7}$$

where b_R is obtained from eq. (6-3-5) and s_R is obtained from either eq. (6-3-1) for a single-test result or from eq. (6-2-1) for a multiple-test result.

The expanded uncertainty in the result at approximately 95% confidence is given by

$$U_{R,95} = 2u_R \tag{6-3-8}$$

where the use of the factor of 2 assumes sufficiently large degrees of freedom for the 95% confidence level (i.e., t_{95} = 2). This factor can be modified as appropriate for other confidence levels and small degrees of freedom, as discussed in Nonmandatory Appendix B.

The interval within which the true result should lie with a 95% level of confidence is given as $\overline{R} \pm U_{R,95}$.

The methodologies for including correlated systematic errors and nonsymmetric systematic uncertainties are covered in subsections 7-1 and 7-2, respectively.

6-4 COMBINED STANDARD UNCERTAINTY AND UNCERTAINTY COVERAGE INTERVAL FOR A RESULT [MONTE CARLO METHOD OF PROPAGATION (MCM)]

With high-speed computing capabilities, the MCM has become popular for determining test result uncertainty using test input variables and their associated uncertainties [10]. The Joint Committee for Guides in Metrology (JCGM) published a supplement [3] to the GUM [2] presenting the MCM for uncertainty analysis. The process is a random sampling from assumed distributions for each error source to estimate the distribution of the determined result.

6-4.1 Single Result at a Given Test Condition

Figure 6-4.1-1, drawn from Coleman and Steele [10], presents the steps for a Monte Carlo process in a flowchart format. The flowchart shows the process for a single test result that is a function of two parameters, but the methodology can be expanded to any number of input parameters. The DRE used can be an analytical expression, a computer data reduction program, or a simulation.

For each input parameter, the measured average, $\overline{X}$, is used as an estimate of the true value of the variable. The random standard uncertainty, $s_{\overline{X}}$, and the elemental systematic standard uncertainties, $b_{\overline{X}_k}$, are input for each parameter along with an assumed distribution for each error source. A Gaussian distribution is appropriate for the random errors, unless the number of measurements is small enough for a t distribution to be used. For systematic errors, engineering judgment is used to assume a distribution. If the possible systematic error for an error source is likely to be zero but has an equal probability to be positive or negative, then a Gaussian distribution can be used. However, if the systematic error is likely to be zero but has finite upper and lower limits, then a triangular distribution would be appropriate. If the systematic error for a given source has finite upper and lower bounds but is equally likely to be a value between these limits, then a rectangular distribution can be used.

Each error distribution is randomly sampled to obtain an error value, and these are added to the estimated true values to obtain current values of the parameters. The results are then calculated. This sampling process is repeated M times to obtain a distribution for the test result. The standard deviation of this distribution, s_{MCM}, is the estimate of the combined standard uncertainty of the result, u_R. The number of the samples required, M, is made on the convergence of s_{MCM}. Periodic checks should be made of the value of s_{MCM} during the Monte Carlo sampling process. Convergence is a matter or judgment, but a value of s_{MCM} that has converged to within 1% to 5% is usually a good approximation of the combined standard uncertainty of the result, u_R.

Note that in Figure 6-4.1-1, the correlated systematic errors are handled directly by assigning the same error value for a common error source, β_3, to each parameter for each iteration. This procedure can be used for all correlated systematic errors. If there are correlated random errors, then the direct approach for multiple results given herein should be used. The procedure shown in Figure 6-4.1-1 can also be used to handle nonsymmetric systematic errors. A distribution is chosen that has the upper and lower standard uncertainty limits as its bounds with zero at the appropriate place between them.

6-4.2 Multiple Results at a Given Test Condition

Figure 6-4.2-1 from Coleman and Steele [10] presents the flowchart for the Monte Carlo process when the random standard uncertainty for the result is estimated directly using eq. (6-2-1). A distribution is assumed for this error source, and the Monte Carlo method follows the same process as described for a single test result.

Figure 6-4.1-1 Monte Carlo Method for Uncertainty Propagation for a Single Test Result

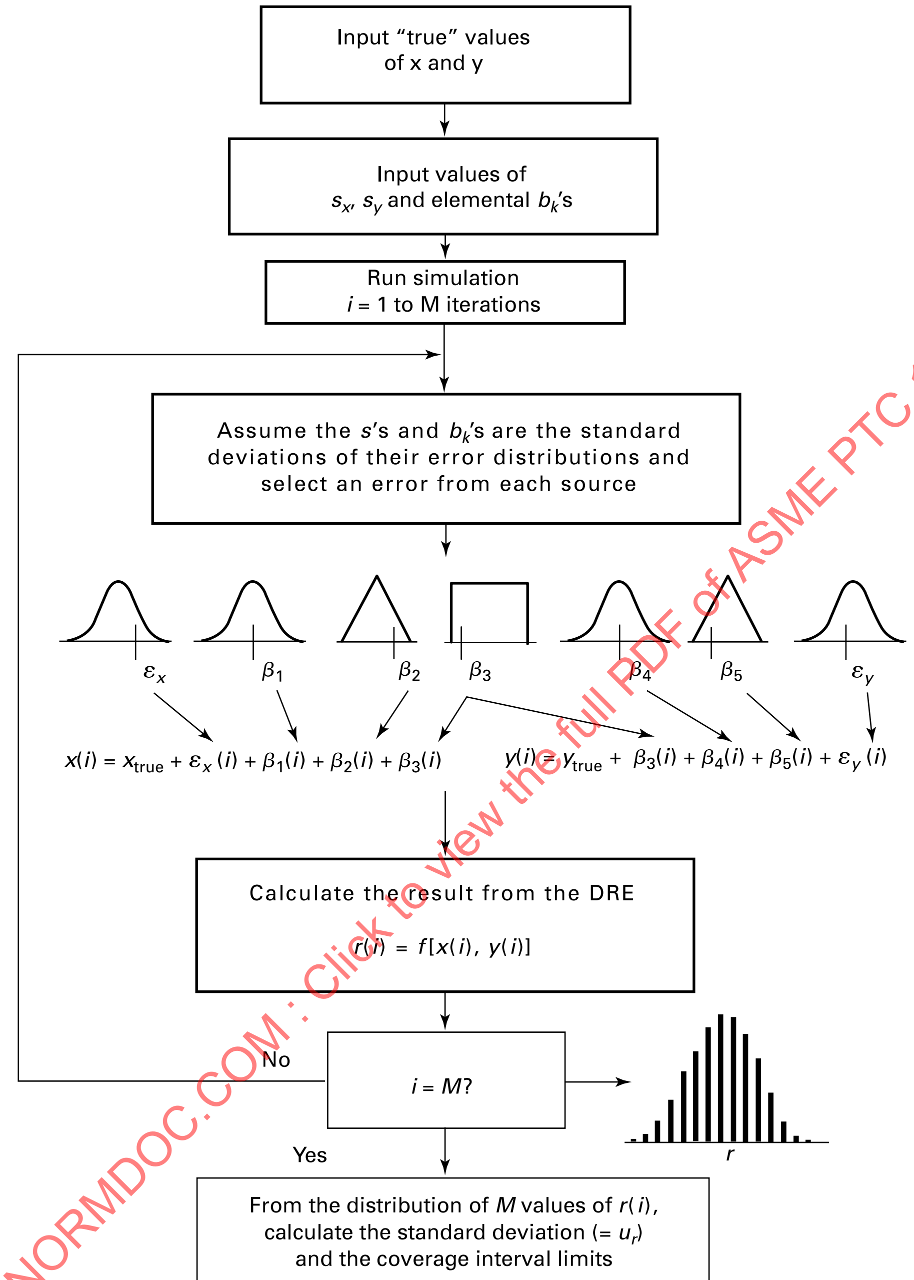

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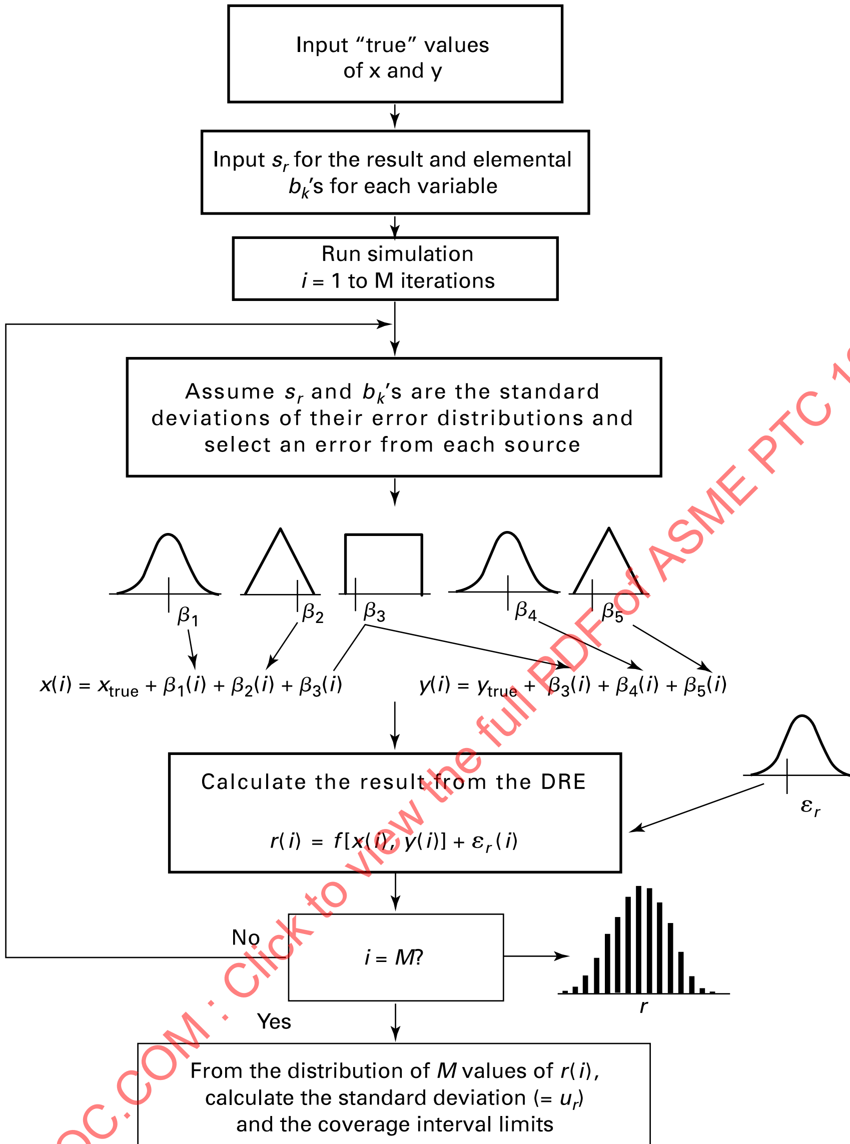
Figure 6-4.2-1 Monte Carlo Method for Uncertainty Propagation for Multiple Results



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6-4.3 Coverage Interval at a Given Level of Confidence

The result distributions from the MCM procedure shown in Figures 6-4.1-1 and 6-4.2-1 are used to find the upper and lower bounds for a 95% coverage interval. The M results are sorted from the lowest to highest values. The bounds on the uncertainty interval are r_{low} and r_{high} where r_{low} is the 0.025M result value and r_{high} is the 0.975M value (i.e., if M is 1 000, then the bounds are the 25th and 975th value). If the 0.025M and 0.975M numbers are not integers, then ½ is added to each and the integer part of the number is used to determine the coverage interval limits. If uncertainty limits are desired, then the resulting nominal value can be used with r_{low} and r_{high} to determine U_r^- and U_r^+ as

$$U_r^- = r(X_1, X_2, \ldots, X_J) - r_{low}$$

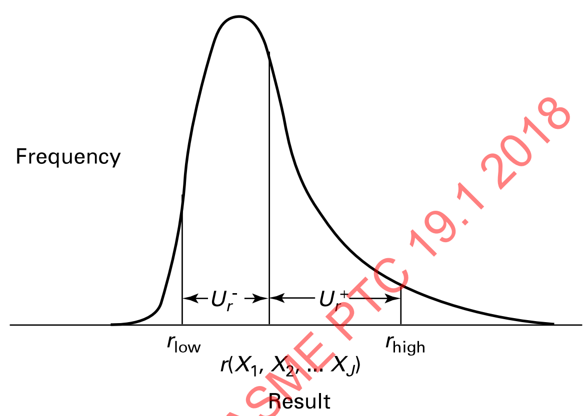
and

$$U_r^+ = r_{high} - r(X_1, X_2, \ldots, X_J)$$

This is illustrated in Figure 6-4.3-1.

For another level of coverage, different multiples of M are used (i.e., at a 90% coverage, the result values 0.05M and 0.95M are used for r_{low} and r_{high}).

Figure 6-4.3-1 Probabilistically Symmetric Coverage Interval



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Section 7
Additional Uncertainty Considerations

7-1 CORRELATED SYSTEMATIC ERRORS (USING TSM PROPAGATION)

This Section documents how to calculate the uncertainty considering correlated sources of error using TSM propagation.

7-1.1 Correlated Systematic Errors

The expressions for the systematic standard uncertainty of the result may assume that the systematic standard uncertainties in each measurand are independent of one another. However, as indicated in para. 6-3.2, there are many situations where the systematic errors in the measurand quantities may not be independent. Examples include calibrating different instruments against the same standard or using the same instruments to make different measurements. Some of these systematic errors are said to be correlated, and these nonindependent errors must be considered in the determination of the systematic standard uncertainty of the result.

Consider a situation where the result, R, is determined from three measurands ($\overline{X}_1$, $\overline{X}_2$, $\overline{X}_3$) that have correlated systematic errors. The result is calculated as

$$R = f(\overline{X}_1, \overline{X}_2, \overline{X}_3) \tag{7-1-1}$$

and the absolute systematic standard uncertainty of the result is given as

$$b_R = \left[\begin{array}{c} \left(\theta_1 b_{\overline{X}_1}\right)^2 + \left(\theta_2 b_{\overline{X}_2}\right)^2 + \left(\theta_3 b_{\overline{X}_3}\right)^2 \\ + 2\theta_1\theta_2 b_{\overline{X}_1\overline{X}_2} + 2\theta_1\theta_3 b_{\overline{X}_1\overline{X}_3} + 2\theta_2\theta_3 b_{\overline{X}_2\overline{X}_3} \end{array} \right]^{1/2} \tag{7-1-2}$$

The first three terms under the square root in eq. (7-1-2) do not account for correlation errors, and the last three terms are those that account for the correlation among the systematic standard errors in $\overline{X}_1$, $\overline{X}_2$, and $\overline{X}_3$. The terms $b_{\overline{X}_i\overline{X}_k}$ are the estimates of the covariance of the systematic errors in $\overline{X}_i$ and $\overline{X}_k$ (see Nonmandatory Appendix B). These terms must be included when systematic standard errors for separate measurands, $\overline{X}_i$ and $\overline{X}_k$, are from the same source, making them correlated; thus, their measurement errors are no longer independent. The units of the correlation terms (covariances), $b_{\overline{X}_i\overline{X}_k}$, are the product of the units of $\overline{X}_i$ and $\overline{X}_k$.

The covariance terms in eq. (7-1-2) must be properly interpreted. Each $b_{\overline{X}_i\overline{X}_k}$ term represents the sum of the products of the portions of $b_{\overline{X}_i}$ and $b_{\overline{X}_k}$ that originate from the same error source and are therefore perfectly correlated [11]. For instance, if elemental systematic standard uncertainties 1 and 2 for measurands 2 and 3 were from a common error source, then $b_{\overline{X}_2\overline{X}_3}$ would be determined as

$$b_{\overline{X}_2\overline{X}_3} = b_{\overline{X}_{2_1}} b_{\overline{X}_{3_1}} + b_{\overline{X}_{2_2}} b_{\overline{X}_{3_2}} \tag{7-1-4}$$

The example in eq. (7-1-2) can be expanded to any number of measurands by including the term for each pair of measurands that has correlated systematic errors. Therefore, the general form of eq. (7-1-2) is

$$b_R{}^2 = \sum_{i=1}^{I} (\theta_i b_i)^2 + 2\sum_{i=1}^{I-1}\sum_{k=i+1}^{I} \theta_i\theta_k b_{ik} \tag{7-1-4}$$

where

b_i = systematic standard uncertainty in the i^{th} measurand

b_{ik} = covariance between the systematic standard uncertainties for the i^{th} and k^{th} measurands, calculated as follows:

$$b_{ik} = \sum_{l=1}^{L} b_{i_l} b_{k_l} \tag{7-1-5}$$

I = an index

i = number of distinct measurands

i and k = indexes indicating the i^{th} and k^{th} measurands

L = number of common (correlated) error sources

θ = sensitivity coefficients

7-1.2 Examples

7-1.2.1 Example 1. The use of back-to-back tests is an excellent method to reduce the systematic standard uncertainty when comparing two or more designs. This method is a special case of correlated systematic standard uncertainties. Consider a burst test for an improved container design. The improvement in the design can be expressed as the fraction

Table 7-1.2-1 Burst Pressures

Back-to-Back Burst Test Design	Base Design, P_b, 10^6 Pa	Improved Design, P_n, 10^6 Pa	Systematic Standard Uncertainty, b_P, 10^6 Pa
Program 1 [Note (1)]			
Meter #1	40.0	...	0.2
Meter #2	...	52.0	0.2
Program 2 [Note (2)]			
Meter #3	42.0	54.7	0.5

NOTES:

(1) Program 1 (no correlated systematic standard uncertainties):

$$R = \frac{52.0}{40.0} = 1.30$$

$$\theta_b = \frac{-R}{P_b} = -0.0325\left(10^6\text{Pa}\right)^{-1}$$

$$\theta_n = \frac{R}{P_n} = 0.0250\left(10^6\text{Pa}\right)^{-1}$$

$$b_R^2 = \left[(-0.0325)\left(10^6\text{Pa}\right)^{-1}(0.2)(10^6\text{Pa})\right]^2 + \left[(0.0250)\left(10^6\text{Pa}\right)^{-1}(0.2)(10^6\text{Pa})\right]^2$$

$b_R = 0.0082$

(2) Program 2 (correlated systematic standard uncertainties):

$$R = \frac{54.7}{42.0} = 1.30$$

$$\theta_b = -0.0310\left(10^6\text{Pa}\right)^{-1}$$

$$\theta_n = 0.0238\left(10^6\text{Pa}\right)^{-1}$$

$$b_R^2 = \left[(-0.0310)\left(10^6\text{Pa}\right)^{-1}(0.5)(10^6\text{Pa})\right]^2 + \left[(0.0238)\left(10^6\text{Pa}\right)^{-1}(0.5)(10^6\text{Pa})\right]^2 + 2(-0.0310)\left(10^6\text{Pa}\right)^{-1}(0.0238)\left(10^6\text{Pa}\right)^{-1} \times (0.5)(10^6\text{Pa})(0.5)(10^6\text{Pa})$$

$b_R = 0.0036$

$$R = \frac{P_n}{P_b}$$

where

P_b = the burst pressure of the original or base design

P_n = the burst pressure of the new design

Table 7-1.2-1 provides burst tests for two different programs. In the first test program, different pressure transducers were used in the tests on the two designs. There were no correlated systematic standard uncertainties common between these two transducers. In the second program, the same pressure transducer was used for both tests; therefore, the systematic error was the same and was correlated for the two test measurements.

This example demonstrates the strength of the back-to-back testing technique using the same instrumentation. Even though the pressure transducer in Program 2 had a systematic standard uncertainty of more than twice that of the transducers in Program 1, the systematic standard uncertainty of the result for program 2 was less than half of that for Program 1. In such cases, the random standard uncertainty of the result may be dominant and must include all sources that vary between the improved design and the base design cases.

7-1.2.2 Example 2. Consider the piping arrangement shown in Figure 7-1.2-1, which has four flowmeters. From conservation of mass, a balance check would yield

$$z = m_4 - m_1 - m_2 - m_3 = 0$$

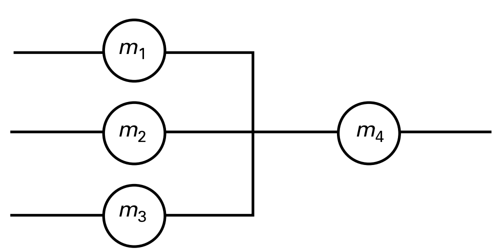
If the errors in the flow-rate measurements are predominantly systematic, then for the balance check to be satisfied the absolute value of z must be less than or equal to the uncertainty in z:

$$|z| \le 2b_z$$

Note that this relationship assumes the degrees of freedom in b_z is greater than or equal to 30.

Equation (7-1-4), repeated herein, may be used to derive eq. (7-1-6) for calculating the systematic uncertainty in the parameter z:

Figure 7-1.2-1 Piping Arrangement With Four Flowmeters



$$b_z{}^2 = \sum_{i=1}^{I} \left(\theta_{m_i} b_{m_i}\right)^2 + \sum_{i=1}^{I-1} \sum_{k=i+1}^{I} \theta_{m_i} \theta_{m_k} b_{m_i m_k} \quad \text{(7-1-4 repeated)}$$

$$\begin{aligned} b_z = \Big[& \left(\theta_{m_1} b_{m_1}\right)^2 + \left(\theta_{m_2} b_{m_2}\right)^2 + \left(\theta_{m_3} b_{m_3}\right)^2 \\ & + \left(\theta_{m_4} b_{m_4}\right)^2 + 2\theta_{m_1}\theta_{m_2} b_{m_1} b_{m_2} \\ & + 2\theta_{m_1}\theta_{m_3} b_{m_1} b_{m_3} + 2\theta_{m_1}\theta_{m_4} b_{m_1} b_{m_4} \\ & + 2\theta_{m_2}\theta_{m_3} b_{m_2} b_{m_3} + 2\theta_{m_2}\theta_{m_4} b_{m_2} b_{m_4} \\ & + 2\theta_{m_3}\theta_{m_4} b_{m_3} b_{m_4} \Big]^{1/2} \end{aligned} \quad \text{(7-1-6)}$$

Note that for this example, the partial derivatives for eq. (7-1-6) are

$$\theta_{m_1} = \theta_{m_2} = \theta_{m_3} = -1$$

and

$$\theta_{m_4} = 1$$

In order to illustrate the effect of correlated sources of error, consider the following cases, where the dominant systematic errors are from the calibration standard and the calibration curve-fit. The calibration standard systematic standard uncertainty for each of the three small flowmeters is ±1.5 kg/h, and ±4.5 kg/h for the large flowmeter. The curve-fit systematic standard uncertainty for each meter is ±0.5 kg/h.

7-1.2.2.1 Case 1: Each Flowmeter Is Calibrated Against a Different Standard. In Case 1, all sources of systematic errors are uncorrelated. The systematic standard uncertainty for the three small flowmeters in Figure 7-1.2-1 is determined as

$$\begin{aligned} b_{m_i}(i = 1, 2, 3) &= \pm\left[(1.5\,\text{kg/h})^2 + (0.5\,\text{kg/h})^2\right]^{1/2} \\ &= \pm 1.58\,\text{kg/h} \end{aligned}$$

and the systematic standard uncertainty for the large flowmeter is calculated as

$$b_{m_4} = \pm\left[(4.5\,\text{kg/h})^2 + (0.5\,\text{kg/h})^2\right]^{1/2} = \pm 4.53\,\text{kg/h}$$

Having no correlated systematic errors causes the covariance between systematic errors ($b_{m_1}b_{m_2}$, $b_{m_1}b_{m_3}$, $b_{m_1}b_{m_4}$, $b_{m_2}b_{m_3}$, $b_{m_2}b_{m_4}$, and $b_{m_3}b_{m_4}$) to be zero. Using eq. (7-1-6), the systematic standard uncertainty for z then becomes as follows:

$$\begin{aligned} b_z = \pm\Big[& \left(\theta_{m_1} b_{m_1}\right)^2 + \left(\theta_{m_2} b_{m_2}\right)^2 + \left(\theta_{m_3} b_{m_3}\right)^2 \\ & + \left(\theta_{m_4} b_{m_4}\right)^2 \Big]^{1/2} \end{aligned}$$

or

$$\begin{aligned} b_z &= \pm\left[\left(b_{m_1}\right)^2 + \left(b_{m_2}\right)^2 + \left(b_{m_3}\right)^2 + \left(b_{m_4}\right)^2\right]^{1/2} \\ &= \pm 5.29\,\text{kg/h} \end{aligned}$$

The condition of conservation of mass will be validated provided the following is satisfied:

$$|z| \leq 2b_z = \pm\, 10.6\,\text{kg/h}$$

7-1.2.2.2 Case 2: Flowmeters 1, 2, and 3 Are Calibrated Against the Same Standard, and Flowmeter 4 Is Calibrated Against a Different Standard. In Case 2, the three small flowmeters in Figure 7-1.2-1 are calibrated against the same standard. This causes any systematic error for this common standard to become correlated for these three meters. The systematic standard uncertainty from their curve-fits, however, is not correlated because it is due to the random scatter in the calibration line. The final standard uncertainty in z is obtained as

$$b_{m_1} = b_{m_2} = b_{m_3} = \pm\, 1.58\,\text{kg/h}$$

$$b_{m_4} = \pm\, 4.53\,\text{kg/h}$$

and

$$b_{m_1 m_2} = b_{m_1 m_3} = b_{m_2 m_3} = \pm\, (1.5\,\text{kg/h})(1.5\,\text{kg/h})$$

Using eq. (7-1-6) with three of four measurands having correlated systematic errors causes the systematic standard uncertainty for z to become

$$\begin{aligned} b_z = \Big[& \left(\theta_{m_1} b_{m_1}\right)^2 + \left(\theta_{m_2} b_{m_2}\right)^2 + \left(\theta_{m_3} b_{m_3}\right)^2 \\ & + \left(\theta_{m_4} b_{m_4}\right)^2 + 2\theta_{m_1}\theta_{m_2} b_{m_1 m_2} \\ & + 2\theta_{m_1}\theta_{m_3} b_{m_1 m_3} + 2\theta_{m_2}\theta_{m_3} b_{m_2 m_3} \Big]^{1/2} \end{aligned}$$

or

$$\begin{aligned} b_z &= \pm\Big[\left(b_{m_1}\right)^2 + \left(b_{m_2}\right)^2 + \left(b_{m_3}\right)^2 + \left(b_{m_4}\right)^2 \\ &\quad + 2b_{m_1 m_2} + 2b_{m_1 m_3} + 2b_{m_2 m_3}\Big]^{1/2} \\ b_z &= \pm 6.4\,\text{kg/h} \end{aligned}$$

The condition of conservation of mass will be validated provided the following is satisfied:

$$|z| \leq 2b_z = \pm\ 12.9\ \text{kg/h}$$

Note that in Case 2, the signs for all the correlated terms are positive because all of the derivatives of z with respect to m_1, m_2, and m_3 are negative. If flowmeters 1, 2, and 3 are calibrated against the same standard, and flowmeter 4 is calibrated against a different standard, the systematic standard uncertainty for z is larger than if all the meters had been calibrated against different standards (see Case 1).

7-1.2.2.3 Case 3: Flowmeters 1, 2, 3, and 4 Are Calibrated Against the Same Standard That Has an Uncertainty Expressed as Percent of Reading. Example 2 began by stating the calibration standard systematic standard uncertainty for each flow meter was ± 1.5 kg/h for the three small meters and ± 4.5 kg/h for the large meter. In Case 3, each of the four flowmeters in Figure 7-1.2-1 is calibrated against the same standard that has specified uncertainty as a percent of the flow rate.

The sketch of flowmeter arrangement for this example shows that meters 1, 2, and 3 are parallel and sum to the flow that is sensed by meter 4. This suggests that, ideally, meters 1, 2, and 3 each provides about one-third of the total flow that is sensed by meter 4. Notice that the common systematic source of uncertainty for meters 1, 2, and 3 is given as 1.5, which is exactly one-third of the common systematic source of uncertainty for meter 4. This proportionality in the systematic uncertainties for the four meters is a result of the systematic uncertainty in the common standard that is used for all four meters being expressed as a percent of reading.

$$b_{m_1} = b_{m_2} = b_{m_3} = \pm\ 1.58\ \text{kg/h}$$

and

$$b_{m_4} = \pm\ 4.53\ \text{kg/h}$$

with

$$b_{m_1m_2} = b_{m_1m_3} = b_{m_2m_3} = \pm\ (1.5\ \text{kg/h})(1.5\ \text{kg/h})$$

and

$$b_{m_1m_4} = b_{m_2m_4} = b_{m_3m_4} = \pm\ (1.5\ \text{kg/h})(4.5\ \text{kg/h})$$

Using eq. (7-1-6) while considering that all four measurands have correlated systematic errors causes the systematic standard uncertainty for z to then become

$$\begin{aligned} b_z = \Big[&\left(\theta_{m_1}b_{m_1}\right)^2 + \left(\theta_{m_2}b_{m_2}\right)^2 + \left(\theta_{m_3}b_{m_3}\right)^2 + \left(\theta_{m_4}b_{m_4}\right)^2 \\ &+ 2\theta_{m_1}\theta_{m_2}b_{m_1}b_{m_2} + 2\theta_{m_1}\theta_{m_3}b_{m_1}b_{m_3} \\ &+ 2\theta_{m_1}\theta_{m_4}b_{m_1}b_{m_4} + 2\theta_{m_2}\theta_{m_3}b_{m_2}b_{m_3} \\ &+ 2\theta_{m_2}\theta_{m_4}b_{m_2}b_{m_4} + 2\theta_{m_3}\theta_{m_4}b_{m_3}b_{m_4}\Big]^{1/2} \end{aligned}$$

or

$$\begin{aligned} b_z &= \pm\Big[\left(b_{m_1}\right)^2 + \left(b_{m_2}\right)^2 + \left(b_{m_3}\right)^2 + \left(b_{m_4}\right)^2 \\ &\quad + 2b_{m_1m_2} + 2b_{m_1m_3} - 2b_{m_1m_4} + 2b_{m_2m_3} \\ &\quad - 2b_{m_2m_4} - 2b_{m_3m_4}\Big]^{1/2} \\ b_z &= \pm 1.0\ \text{kg/h} \end{aligned}$$

The condition of conservation of mass will be validated provided the following is satisfied:

$$|z| \leq 2b_z = \pm\ 2.0\ \text{kg/h}$$

Note the signs for each of the correlated terms.

7-1.2.2.4 Case 4: Flowmeters 1, 2, 3, and 4 Are Calibrated Against the Same Standard That Has an Uncertainty Expressed as Percent of Full Scale. In Case 4, each of the four flowmeters in Figure 7-1.2-1 is calibrated against the same standard; however, the systematic standard uncertainty from the standard is a fixed value of ±4.5 kg/h across all flow rates. This implies that the calibration standard systematic standard uncertainty is expressed as a percent of full scale. The systematic standard uncertainty for each flowmeter then becomes

$$\begin{aligned} b_{m_1} = b_{m_2} = b_{m_3} = b_{m_4} &= \pm\left[(4.5)^2 + (0.5)^2\right]^{1/2} \\ &= \pm 4.53\ \text{kg/h} \end{aligned}$$

with

$$\begin{aligned} b_{m_1m_2} &= b_{m_1m_3} = b_{m_1m_4} = b_{m_2m_3} = b_{m_2m_4} = b_{m_3m_4} \\ &= \pm\ (4.5\ \text{kg/h})\ (4.5\ \text{kg/h}) \end{aligned}$$

The systematic standard uncertainty in z per eq. (7-1-6) is calculated as follows:

$$\begin{aligned} b_z = \Big[&\left(\theta_{m_1}b_{m_1}\right)^2 + \left(\theta_{m_2}b_{m_2}\right)^2 + \left(\theta_{m_3}b_{m_3}\right)^2 + \left(\theta_{m_4}b_{m_4}\right)^2 \\ &+2\theta_{m_1}\theta_{m_2}b_{m_1}b_{m_2} + 2\theta_{m_1}\theta_{m_3}b_{m_1}b_{m_3} \\ &+2\theta_{m_1}\theta_{m_4}b_{m_1}b_{m_4} + 2\theta_{m_2}\theta_{m_3}b_{m_2}b_{m_3} \\ &+2\theta_{m_2}\theta_{m_4}b_{m_2}b_{m_4} + 2\theta_{m_3}\theta_{m_4}b_{m_3}b_{m_4}\Big]^{1/2} \end{aligned}$$

or

$$\begin{aligned} b_z &= \pm\Big[\left(b_{m_1}\right)^2 + \left(b_{m_2}\right)^2 + \left(b_{m_3}\right)^2 + \left(b_{m_4}\right)^2 \\ &\quad +2b_{m_1m_2} + 2b_{m_1m_3} - 2b_{m_1m_4} + 2b_{m_2m_3} \\ &\quad -2b_{m_2m_4} - 2b_{m_3m_4}\Big]^{1/2} \\ b_z &= \pm 9.06\ \text{kg/h} \end{aligned}$$

The condition of conservation of mass will be validated provided the following is satisfied:

$$|z| \leq 2b_z = \pm\ 18.1\ \text{kg/h}$$

Note the signs for each of the correlated terms.

The following conclusions can be made from these four cases:

(a) calibrating all flowmeters against a common standard that had a percent of reading systemic uncertainty yielded a systematic standard uncertainty in the result (z) that was less than the systematic standard uncertainty for any give flowmeter (Case 3)

(b) correlated systematic errors between measurands do not necessarily reduce the systematic uncertainty in the result (Case 2 and Case 4)

In general, the ability of correlated systematic errors to decrease, increase, or have no effect on the systematic standard uncertainty of the result depends on the form of the DRE and on which measurands have correlated systematic errors.

Once again, the random standard uncertainty in z may dominate the combined standard uncertainty and must be carefully determined and included in the expanded uncertainty determination of z.

7-2 NONSYMMETRIC SYSTEMATIC UNCERTAINTY (TSM PROPAGATION)

In some experiments, physical models (e.g., radiative heat transfer models for temperature measurement) may be used to essentially replace the asymmetric uncertainties with symmetric uncertainties in additional experimental variables. If this can be done then it should be; if not, then the method of para. 7-2.1 should be used.

7-2.1 Nonsymmetric Systematic Uncertainty Interval for a True Value

This paragraph presents a method for determining nonsymmetric uncertainty intervals using TSM propagation [10, 12].

If the distribution of the systematic error associated with a variable is nonsymmetrical, then the overall uncertainty interval for the unknown true value will not be centered on the measured value of the variable. The following procedure can be used to construct a nonsymmetric uncertainty interval for the unknown true value of the quantity being measured.

The procedure is based on first establishing a lower limit (LL) and an upper limit (UL) for the possible systematic error distribution. For instance, the measurement of the temperature of a hot gas stream flowing in a pipe may have an error due to radiative heat transfer between the measurement transducer and the pipe wall. An estimate of the effect of the radiation error might be that the true temperature could be as much as 2°C less than the transducer measurement (LL = 2°C) and could be as much as 20°C above the measurement (UL = 20°C).

The next step is to assign the distribution for the possible nonsymmetric systematic errors. One approach, as demonstrated in Figure 7-2.1-1, is to assume that $\overline{X}-\mathrm{LL}$ and $\overline{X}+\mathrm{UL}$ represent the plus and minus bounds for a 95% confidence interval for a Gaussian distribution, with the probable systematic error at the midpoint of the distribution (see Figure 7-2.1-1).

Another approach is to assume an equal probability of occurrence for any error value between $\overline{X}-\mathrm{LL}$ and $\overline{X}+\mathrm{UL}$. In this case, a rectangular error distribution would be appropriate, as shown in Figure 7-2.1-2. A third approach would allow for the most probable error to be at any value between $\overline{X}-\mathrm{LL}$ and $\overline{X}+\mathrm{UL}$. In this case, a triangular distribution would be used with a user-defined most probable limit (MPL), as shown in Figure 7-2.1-3. Note that if the most probable error is less than $\overline{X}$, then MPL will be negative. For the temperature measurement example above, if the most likely value of the true temperature is 18°C above the measurement, then MPL = 18.

The following procedure gives the option of choosing a Gaussian, rectangular, or triangular distribution for the nonsymmetric error:

(a) specify the lower limit (LL), upper limit (UL), and, if appropriate, most probable limit (MPL) for the nonsymmetric error distribution.

(b) define the offset, q, as the difference between the mean of the distribution specified in (a) and the measured value. The expressions for calculating the offset for each of the three distribution types are given in Table 7-2.1-1. Note that the expression for the Gaussian and rectangular distributions are the same because the means are in the centers of these two distributions.

Note that q can be positive or negative depending on the relative values of UL, LL, and MPL (where MPL can be negative as described in this paragraph). If $\overline{X}$ is greater than the mean of the distribution, q will be negative, and if $\overline{X}$ is less than the mean of the distribution, q will be positive.

(c) calculate $b_{\overline{X}_{ns}}$, the systematic standard uncertainty for the nonsymmetric error distribution, using the appropriate expression in Table 7-2.1-2.

(d) combine the systematic standard uncertainty with the others for the measurement to obtain $b_{\overline{X}}$ using eq. (5-2-1).

(e) calculate $u_{\overline{X}}$, the combined standard uncertainty for the measurement, using the standard formula

$$u_{\overline{X}} = \sqrt{b_{\overline{X}}^2 + s_{\overline{X}}^2}$$

(f) calculate U_{95}, the expanded uncertainty for the measurement, using

$$U_{95} = 2u_{\overline{X}} = 2\sqrt{b_{\overline{X}}^2 + s_{\overline{X}}^2} \tag{7-2-1}$$

Figure 7-2.1-1 Gaussian Distribution for Nonsymmetric Systematic Errors

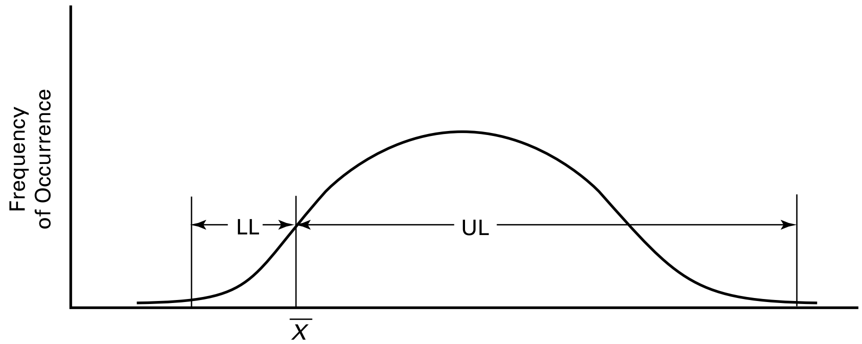


Figure 7-2.1-2 Rectangular Distribution for Nonsymmetric Systematic Errors

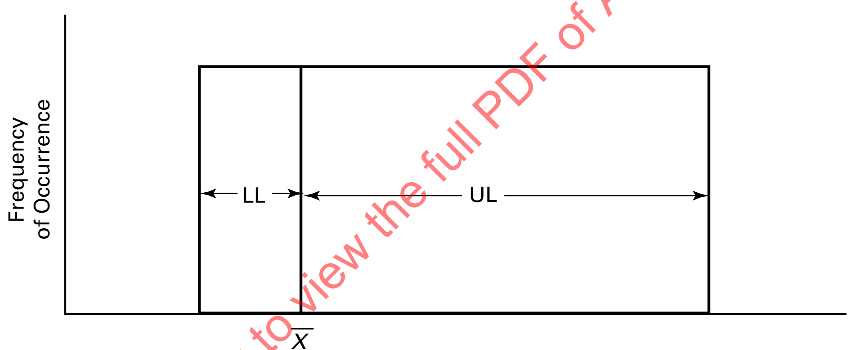


Figure 7-2.1-3 Triangular Distribution for Nonsymmetric Systematic Errors

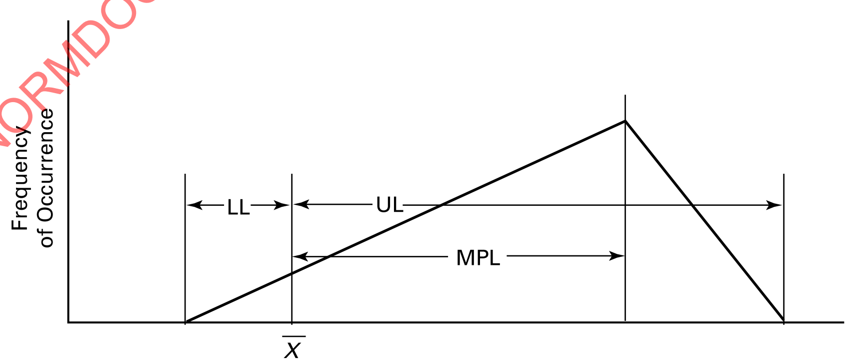


Table 7-2.1-1 Expressions for *q* for the Gaussian, Rectangular, and Triangular Distributions in Figures 7-2.1-1 through 7-2.1-3

Distribution	*q*
Gaussian	$\frac{\text{UL} - \text{LL}}{2}$
Rectangular	$\frac{\text{UL} - \text{LL}}{2}$
Triangular	$\frac{\text{UL} - \text{LL} + \text{MPL}}{3}$

The calculation is based on the assumption that the degrees of freedom of the combined standard uncertainty are large. (For small degrees of freedom, see Nonmandatory Appendix B.)

Note that if the nonsymmetric systematic uncertainty is non-Gaussian and dominates the uncertainty determination, the Central Limit Theorem (see Nonmandatory Appendix C) may not apply, and eq. (7-2-1) would not be appropriate to determine the expanded uncertainty. In this case, a Monte Carlo technique would be appropriate to determine a 95% coverage interval [10]. The upper and lower limits (based on $\overline{X} + q$) of this nonsymmetrical interval from the Monte Carlo technique would then be combined with q as shown in (g) and (h) to determine the nonsymmetric limits for $\overline{X}$.

(g) calculate an approximate 95% confidence interval for the true value using

$$[\overline{X} + q] \pm U_{95} \tag{7-2-2}$$

(h) express the final result as an asymmetric 95% confidence interval for the true value with the lower limit given by

$$\overline{X}_{\text{lower limit}} = \overline{X} + q - U_{95} = \overline{X} - U^{-} \tag{7-2-3}$$

and the upper limit given by

$$\overline{X}_{\text{upper limit}} = \overline{X} + q + U_{95} = \overline{X} + U^{+} \tag{7-2-4}$$

where

$U^{-} = U_{95} - q$
$U^{+} = U_{95} + q$

7-2.2 Example 1

Suppose a thermocouple is being used to measure the temperature of a gas stream, but the user of the thermocouple believes there may be a tendency for the thermocouple to provide a temperature reading lower than the actual gas temperature. Due to insufficient information about the gas stream flow rate, the user is not able to properly correct the thermocouple reading for these effects, but wishes to account for them in an uncertainty analysis. The decision is made to account for heat transfer effects through the use of a triangular distribution.

From a sample of more than 30 readings using the thermocouple, the user finds that $\overline{X}$ = 534.7°C and $s_{\overline{X}}$ = 2.4°C. If the user believes that the true gas temperature may be between 1°C lower and 10°C higher than $\overline{X}$ due to radiation effects, then a nonsymmetric confidence interval accounting for this nonsymmetric systematic uncertainty may be computed as follows:

(a) specify an interval for the systematic error in question. In this case, the user of the thermocouple believes that the true gas temperature falls within a range of 1°C lower and 10°C higher than the average measured with the thermocouple, $\overline{X}$ = 534.7°C, with the most likely true temperature being 8°C higher. So for this example, a triangular distribution is used with LL = 1°C, UL = 10°C, and MPL = 8°C. This distribution is illustrated graphically in Figure 7-2.2-1.

(b) determine q, the difference between the mean of the distribution specified in (a) and the value measured with the thermocouple. In this case

$$q = \frac{10 - 1 + 8}{3} = 5.7°\text{C}$$

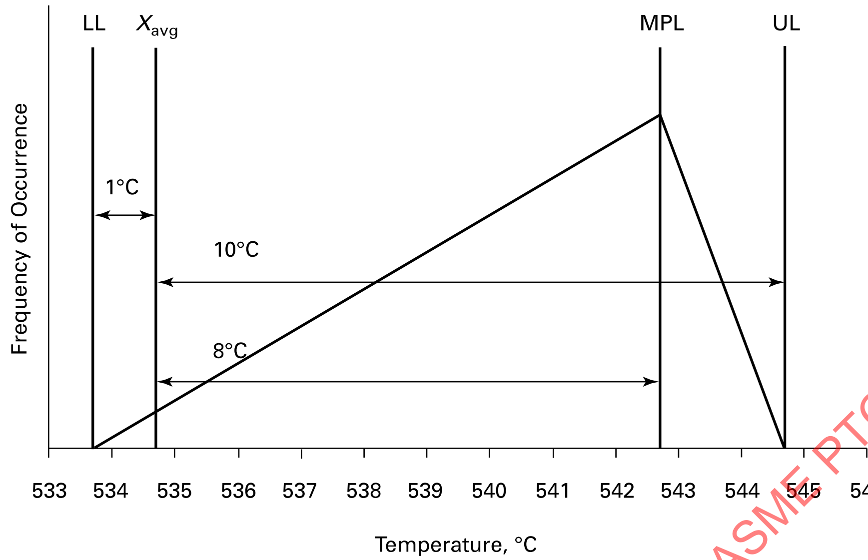
(c) calculate $b_{\overline{X}_{ns}}$, the systematic standard uncertainty for the nonsymmetric systematic error, as

$$b_{\overline{X}_{ns}} = \left[\frac{(10)^2 + (1)^2 + (8)^2 + (1)(10) + (1)(8) - (10)(8)}{18}\right]^{1/2} = 2.4°\text{C}$$

Table 7-2.1-2 Systematic Standard Uncertainties, $b_{\overline{x}_{ns}}$, for the Gaussian, Rectanglar, and Triangular Distributions in Figures 7-2.1-1 through 7-2.1-3

Distribution	$b_{\overline{X}_{ns}}$
Gaussian	$\frac{\text{UL} + \text{LL}}{4}$
Rectangular	$\frac{\text{UL} + \text{LL}}{2\sqrt{3}}$
Triangular	$\left[\frac{\text{UL}^2 + \text{LL}^2 + \text{MPL}^2 + (\text{LL})(\text{UL}) + (\text{LL})(\text{MPL}) - (\text{UL})(\text{MPL})}{18}\right]^{1/2}$

Figure 7-2.2-1 Triangular Distribution of Temperatures



(d) calculate $b_{\overline{X}}$, the systematic standard uncertainty for the measurement. For the example, the nonsymmetric systematic uncertainty is the dominant systematic uncertainty, so

$$b_{\overline{X}} = b_{\overline{X}_{ns}}.$$

(e) calculate $u_{\overline{X}}$, the combined standard uncertainty for the measurement, using the standard formula:

$$u_{\overline{X}} = \sqrt{b_{\overline{X}}^2 + s_{\overline{X}}^2} = \sqrt{(2.4°\text{C})^2 + (2.4°\text{C})^2} = 3.4°\text{C}$$

(f) calculate U_{95}, the expanded uncertainty for the measurement, using

$$U_{95} = 2u_{\overline{X}} = 6.8°\text{C}$$

This calculation is based on the assumption that the degrees of freedom for the combined standard uncertainty are large. (For small degrees of freedom, see Nonmandatory Appendix B.)

(g) calculate an approximate 95% confidence interval for the true value using $[\overline{X} + q] \pm U_{95}$. In this case, this 95% confidence interval is given by

$$[534.7°\text{ C} + 5.7°\text{C}] \pm 6.8°\text{C}$$

(h) calculate

$$U^- = U_{95} - q = 6.8°\text{C} - 5.7°\text{C} = 1.1°\text{C} \text{ and}$$

$$U^+ = U_{95} + q = 6.8°\text{C} + 5.7°\text{C} = 12.4°\text{C}$$

Figure 7-2.2-1 charts the final result, which may be expressed as an asymmetric 95% confidence interval for the true value with the lower limit given by

$$\overline{X}_{\text{lower limit}} = \overline{X} - U^- = 534.7°\text{C} - 1.1°\text{C} = 533.6°\text{C}$$

and the upper limit given by

$$\overline{X}_{\text{upper limit}} = \overline{X} + U^+ = 534.7°\text{C} + 12.4°\text{C} = 547.1°\text{C}$$

7-2.3 Nonsymmetric Systematic Uncertainty Interval for a Derived Result

A nonsysmmetric systematic uncertainty in a measured variable may also result in a nonsysmmetric uncertainty interval for a derived result. The following procedure may be employed for propagating the nonsymmetric uncertainties in a set of measured variables to a derived result:

(a) determine $\overline{X}_i$, $u_{\overline{X}_i}$, and q_i for each average $\overline{X}_i$ that contributes to the determination of the derived result, $r(\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_n)$

(b) determine the offset, q_r, which is defined as $q_r = r(\overline{X}_1 + q_1, \overline{X}_2 + q_2, \ldots, \overline{X}_n + q_n) - r(\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_n)$

(c) determine the sensitivity coefficient, θ_i, for each average $\overline{X}_i$ that contributes to the derived result following standard procedure. If a sensitivity coefficient depends on the values of any averages, i.e., $\theta_i = \theta_i(\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_n)$, then it shoulde be evaluated at the point $(\overline{X}_1 + q_1, \overline{X}_2 + q_2, \ldots, \overline{X}_n + q_n)$

(d) calculate u_r, the combined standard uncertainty for the derived result, using that standard formula

$$u_r = \sqrt{\left[\left(\theta_1 u_{\overline{X}_1}\right)^2 + \left(\theta_2 u_{\overline{X}_2}\right)^2 + \ldots + \left(\theta_n u_{\overline{X}_n}\right)^2\right]} \quad (7\text{-}2\text{-}5)$$

(e) calculate $U_{95,r}$, the expanded uncertainty for the derived result at a 95% confidence level, as $U_{95,r} = 2\mu_r$. This is based on the assumption that the degrees of the freedom are large. For small degrees of freedom, see Nonmandatory Appendix B.

(f) calculate an approximate 95% confidence interval for the derived result using

$$r(\overline{X}_1 + q_1, \overline{X}_2 + q_2, \ldots, \overline{X}_n + q_n) \pm U_{95,r} \tag{7-2-6}$$

(g) express the confidence interval as an asymmetric 95% confidence interval for the derived result as follows:

$$r(\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_n) \pm (U_{95,r} \pm q_r) \tag{7-2-7}$$

where the lower limit on this interval is given by

$$\begin{aligned} r_{\text{lower limit}} &= r(\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_n) \\ &\quad - (U_{95,r} - q_r) = r - U_r^- \end{aligned} \tag{7-2-8}$$

and the upper limit on this interval is given by

$$\begin{aligned} r_{\text{upper limit}} &= r(\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_n) \\ &\quad + (U_{95,r} + q_r) = r + U_r^+ \end{aligned} \tag{7-2-9}$$

with $U_r^- = U_{95,r} - q_r$ and $U_r^+ = U_{95,r} + q_r$

7-2.4 Example 2

Suppose the user of the thermocouple in Example 1 in para. 7-2.2 wishes to use this gas temperature to estimate the speed of sound for the gas using the following relation, $c = [kRT]^{1/2}$, where k, the ratio of specific heats, and R, the gas constant for the gas, are taken to be constant with negligible uncertainty, and T is the measured value of the absolute temperature in this thermocouple. The uncertainty interval for c may be calculated as follows:

(a) determine T, u_T and q_T for the measured variable T. In this case, $T = 807.9K$, $u_T = 3.4K$, and $q_T = 5.7K$

(b) determine the offset, q_c, as follows:

$$\begin{aligned} q_c &= c(T + q_T) - c(T) \\ &= [kR(813.6)]^{1/2} - [kR(807.9K)]^{1/2} \\ &= (kR)^{1/2}(0.100K^{1/2}) \end{aligned}$$

(c) determine the sensitivity coefficient, θ_T, for the measured variable T. In this case, $\theta_T = (\frac{1}{2})(\frac{kR}{T})^{1/2}$. Since this sensitivity coefficient depends on T, it should be evaluated as $T + q_T = 813.6K$, so that here

$$\theta_T = \left(\frac{1}{2}\right)[kR/(813.6K)]^{1/2} = (kR)^{1/2}\left(0.0175K^{-1/2}\right)$$

(d) estimate the combined standard uncertainty for the derived results u_c. In this case,

$$\begin{aligned} u_c &= \left[\left\{(kR)^{1/2}(0.0175K^{-1/2})(3.4K)\right\}^2\right]^{1/2} \\ &= [kR]^{1/2}(0.0596K^{1/2}) \end{aligned}$$

(e) calculate $U_{95,c}$, the expanded uncertainty for the derived result c, at a 95% confidence level, as

$$U_{95,c} = 2u_c = 2[kR]^{1/2}(0.0596K^{1/2}) = [kR]^{1/2}(0.119K^{1/2})$$

This is based on the assumption that the degrees of freedom are large. For small degrees of freedom, see Nonmandatory Appendix B.

(f) compute a 95% confidence interval for the derived result using $c(T + q_T) \pm U_{95,c}$. In this case, this 95% confidence interval is given by

$$(kR)^{1/2}(813.6)^{1/2} \pm (kR)^{1/2}(0.119K)^{1/2}$$

(g) express the final result as an asymmetric 95% confidence interval using

$$c(T) \pm (U_{95,c} \pm q_c)$$

In this case, this 95% confidence interval is given by

$$\begin{aligned} &(kR)^{1/2}(807.9)^{1/2} \pm (kR)^{1/2}(0.119K)^{1/2} \\ &\pm (kR)^{1/2}(0.100K)^{1/2} \end{aligned}$$

$$c_{\text{lower limit}} = (kR)^{1/2}(28.42K)^{1/2} - (kR)^{1/2}(0.019K)^{1/2}$$

and whose upper limit is equal to

$$c_{\text{upper limit}} = (kR)^{1/2}(28.42K)^{1/2} + (kR)^{1/2}(0.219K)^{1/2}$$

In this example, the uncertainty interval for the speed of sound of the gas extends from 0.07% below to 0.77% above the value for the speed of sound assessed using the measured value of the temperature.

7-3 REGRESSION UNCERTAINTY (TSM)

7-3.1 Linear Regression Analysis

Curve-fitting often is used in the calibration process, in the data reduction program, and in the representation of the final test results. Least-squares-regression analysis is the most popular means of curve-fitting. In many cases, the anticipated representation of the data is a straight line, or a simple (first-order) linear regression. In some other cases, the data to be curve-fit can be rectified, or transformed, into linear coordinates [10, 13, 14].

Higher-order linear regressions and other regression methodologies are discussed in detail in ISO/TR 7066-2 and in textbooks [10, 15, 16, 17], as is regression uncertainty when X and Y are functions of other variables [10, 17]. An overview of these topics is provided in Nonmandatory Appendix D.

The random standard uncertainty for the curve-fit will be determined using standard least-squares analysis [10, 14, 16] where the assumption is made that there is no random standard uncertainty in the X values and the random standard uncertainty in the Y values is constant over the range of the curve-fit.

In this Section, only a special case is considered for the systematic standard uncertainty. This special case is where the systematic standard uncertainty for the Y values and/or the X values is a constant (i.e., percent of full scale) and there are no correlated elemental systematic errors between the X and Y values. A more general approach to regression uncertainty is presented in [10] and summarized in Nonmandatory Appendix D, where the methodology applies for variable random standard uncertainties in X and Y, variable systematic standard uncertainties in X and Y, and correlated systematic errors between X and Y.

7-3.2 Least-Squares

For a straight-line, or simple linear regression, the curve-fit expression is

$$\hat{Y} = mX + c \tag{7-3-1}$$

where for N data pairs, X_j, Y_j, the slope m is determined from

$$m = \frac{N \sum_{j=1}^{N} X_j Y_j - \sum_{j=1}^{N} X_j \sum_{j=1}^{N} Y_j}{N \sum_{j=1}^{N} (X_j^2) - \left(\sum_{j=1}^{N} X_j \right)^2} \tag{7-3-2}$$

and the intercept c is determined from

$$c = \frac{\sum_{j=1}^{N} (X_j^2) \sum_{j=1}^{N} Y_j - \sum_{j=1}^{N} X_j \sum_{j=1}^{N} (X_j Y_j)}{N \sum_{j=1}^{N} (X_j^2) - \left(\sum_{j=1}^{N} X_j \right)^2} \tag{7-3-3}$$

The least-squares process essentially provides an average for the data so that the regression expression in eq. (7-3-1) represents the relationship between the mean value of Y and X. This means $\hat{Y}$ is not the average of the Y_j data but the mean Y response from the curve-fit for a given X. Once the slope and intercept are calculated from eqs. (7-3-2) and (7-3-3), these constants can be substituted into eq. (7-3-1) along with several values of X and the resulting straight line can be plotted over the X_j, Y_j data. Since the $\hat{Y}$-versus-X curve is a mean value for the data set, the curve should be a good representation of the data if the simple linear fit is appropriate.

7-3.3 Random Standard Uncertainty for $\hat{Y}$ Determined From Regression Equation

The statistic that defines the standard deviation for a straight-line curve-fit is the standard error of estimate

$$\text{SEE} = \left[\frac{\sum_{j=1}^{N} \left(Y_j - mX_j - c \right)^2}{N - 2} \right]^{1/2} \tag{7-3-4}$$

For a given value of X, the random standard uncertainty associated with the $\hat{Y}$ obtained from the curve-fit [eq. (7-3-1)] is

$$s_{\hat{Y}} = \text{SEE} \left[\frac{1}{N} + \frac{(X - \overline{X})^2}{\sum_{j=1}^{N} \left(X_j - \overline{X} \right)^2} \right]^{1/2} \tag{7-3-5}$$

where

$$\overline{X} = \frac{1}{N} \sum_{j=1}^{N} X_j \tag{7-3-6}$$

It is also assumed that the proper regression expression for the data is a straight line and that the variation of the Y values around the curve-fit results from the random error in the Y measurements.

If there is no random standard uncertainty in the X_j data or the new X values used in the regression equation, the random standard uncertainty $s_{\hat{Y}}$ obtained from eq. (7-3-5) is combined with the systematic standard uncertainty (discussed in para. 7-3-4) using eq. (7-3-7) to obtain the combined standard uncertainty for the $\hat{Y}$ value from the curve-fit. For random standard uncertainty in the X_j or X values, the general approach in this Code or Nonmandatory Appendix D should be used.

7-3.4 Systematic Standard Uncertainty for $\hat{Y}$ Determined From Regression Equation

There can be systematic standard uncertainty, $b_{\hat{Y}}$ respectively, in the Y_j and X_j data. There also can be systematic standard uncertainty in the X value used in the curve-fit to find a $\hat{Y}$ value. This curve-fit X will be called X_{new} to distinguish it from the X_j data points, and the systematic standard uncertainty for X_{new} is $b_{X_{\text{new}}}$. It is very likely that most, and probably all, of the elemental systematic standard uncertainties for each of the Y_j data points are from the same error sources, and are, therefore, correlated. The same is true for the X_j data points. There is also a possibility that the X_{new} values will have systematic standard uncertainties from the same sources as the X_j data, causing these uncertainties to be correlated.

Only constant systematic standard uncertainties for X_j, Y_j, and X_{new} are considered. All of the b_{Y_j} uncertainties are assumed to be completely correlated with each other, and all of the b_{X_j} uncertainties are assumed to be completely correlated with each other. It is assumed there are no common uncertainty sources between Y_j and X_j (no correlation between the b_{Y_j} and b_{X_j} systematic standard uncertainties). Cases are considered where X_{new} has systematic standard uncertainty correlated with that in X_j and where X_{new} has systematic standard uncertainty not correlated with that in X_j.

7-3.4.1 Systematic Standard Uncertainty in Y_j Data. If each of the Y_j data points had the same systematic standard uncertainty, b_Y, then the resulting elemental systematic standard uncertainty for the mean $\hat{Y}$ from the curve-fit [10, 14, 17] is

$$b_{\hat{Y}_1} = b_Y \tag{7-3-7}$$

7-3.4.2 Systematic Standard Uncertainty in X_j Data With No Systematic Standard Uncertainty in X_{new}. If each of the X_j data points has the same systematic standard uncertainty, b_X, and X_{new} has no systematic standard uncertainty, then the resulting elemental systematic standard uncertainty for the mean $\hat{Y}$ from the curve-fit [10, 17] is determined as

$$b_{\hat{Y}_2} = m b_X \tag{7-3-8}$$

This occurs when the regression equation from a set of test data is used later in a design or analysis process where X_{new} might be taken as a value that has no uncertainty.

7-3.4.3 Systematic Standard Uncertainty in X_j Data With Correlated Systematic Standard Uncertainty in X_{new}. If each of the X_j data points had the same systematic standard uncertainty, b_X, and X_{new} has the same systematic standard uncertainty (from the same sources), then the resulting elemental systematic standard uncertainty for the mean $\hat{Y}$ from the curve-fit is zero [10]. This case would occur if the same instruments are used to measure X_{new} as were used to measure X_j. Since all of the systematic standard uncertainties for X_j and X_{new} are correlated, the systematic standard errors are all the same. The effect on the curve-fit is to shift it to the right or left depending on the sign of the errors (the signs and magnitudes of the errors are unknown). This shift has no effect on the value of $\hat{Y}$ obtained from the curve since the shift in X_{new} is the same as the shift in X_j.

7-3.4.4 Systematic Standard Uncertainty in X_j Data With Uncorrelated Systematic Standard Uncertainty in X_{new}. If each of the X_i data points had the same systematic standard uncertainty, b_X, but X_{new} has a different (no common systematic error sources) systematic standard uncertainty, $b_{X_{\text{new}}}$, then the resulting elemental systematic standard uncertainty for the mean $\hat{Y}$ from the curve-fit is

$$b_{\hat{Y}_3} = \left[(m b_X)^2 + \left(m b_{X_{\text{new}}}\right)^2\right]^{1/2} \tag{7-3-9}$$

This would occur if different instruments were used to measure the X_j values and X_{new}.

7-3.4.5 Systematic Standard Uncertainty for $\hat{Y}$. The systematic standard uncertainty for the mean $\hat{Y}$ from the curve-fit will be the appropriate root-sum-square of the $b_{\hat{Y}_j}$ elemental systematic standard uncertainties defined herein and summarized in Table 7-3.4-1.

For systematic standard uncertainty in Y_j only or systematic standard uncertainty in Y_j with correlated systematic standard uncertainty between X_j and X_{new}:

$$b_{\hat{Y}} = b_{\hat{Y}_1} \tag{7-3-10}$$

For systematic standard uncertainty in the Y_j data and the X_j data and no systematic standard uncertainty in X_{new}:

$$b_{\hat{Y}} = \left(b_{\hat{Y}_1}^2 + b_{\hat{Y}_2}^2\right)^{1/2} \tag{7-3-11}$$

For systematic standard uncertainty in the Y_j data and the X_j data and uncorrelated systematic standard uncertainty in X_{new}, the systematic standard uncertainty for the curve-fit value of $\hat{Y}$ is

$$b_{\hat{Y}} = \left(b_{\hat{Y}_1}^2 + b_{\hat{Y}_3}^2\right)^{1/2} \tag{7-3-12}$$

For no systematic standard uncertainty in the Y_j data, systematic standard uncertainty in the X_j data, and no systematic standard uncertainty in X_{new}:

$$b_{\hat{Y}} = b_{\hat{Y}_2} \tag{7-3-13}$$

For no systematic standard uncertainty in the Y_j data, systematic standard uncertainty in the X_j data, and uncorrelated systematic standard uncertainty in X_{new}, the systematic standard uncertainty for the curve-fit value of $\hat{Y}$ is

Table 7-3.4-1 Systematic Standard Uncertainty Components for $\hat{Y}$ Determined From Regression Equation

Components	Equation
Systematic standard uncertainty in Y_j data	$b_{\hat{Y}_1} = b_{Y_1}$
Systematic standard uncertainty in X_j data with no systematic standard uncertainty in X_{new}	$b_{\hat{Y}_2} = m b_X$
Systematic standard uncertainty in X_j data with correlated systematic standard uncertainty in X_{new}	0
Systematic standard uncertainty in X_j data with uncorrelated systematic standard uncertainty in X_{new}	$b_{\hat{Y}_3} = \left[(m b_X)^2 + \left(m b_{X_{\text{new}}}\right)^2\right]^{1/2}$

$$b_{\hat{Y}} = b_{\hat{Y}_3} \quad (7\text{-}3\text{-}14)$$

For no systematic standard uncertainty in the Y_j data and correlated systematic standard uncertainty between X_j and X_{new}:

$$b_{\hat{Y}} = 0 \quad (7\text{-}3\text{-}15)$$

7-3.5 Uncertainty for $\hat{Y}$ From Regression Equation

The total uncertainty in the $\hat{Y}$ obtained from the simple linear regression expression, eq. (7-3-1), is given by eqs. (6-3-7) and (6-3-8) for the case where the degrees of freedom for $\hat{Y}$ are sufficiently large so that $t \approx 2$.

$$U_{\hat{Y}} = 2\left[b_{\hat{Y}}^2 + s_{\hat{Y}}^2\right]^{1/2} \quad (7\text{-}3\text{-}16)$$

Note that the degrees of freedom for $\hat{Y}$ is based on the degrees of freedom for $s_{\hat{Y}}$, which is $N - 2$, and the degrees of freedom for $b_{\hat{Y}}$ (see Nonmandatory Appendix B). The use of the factor $t \approx 2$ will be appropriate in most cases. The uncertainty band $U_{\hat{Y}}$ in eq. (7-3-16) will vary with X (i.e., X_{new}) because of the expression for $s_{\hat{Y}}$ from eq. (7-3.5). As noted in para. 7-3.3, the uncertainty expression in eq. (7-3-16) only applies if there is no random standard uncertainty in X and if the systematic standard uncertainties are percent of full-scale values or are fixed and do not change across the range of the instrument.

Section 8
A Comprehensive Example

This Section is derived from Coleman and Steele [10] and is divided into the following parts to reflect the textbook analysis:

Part 1	Overview			
Part 2	Generic Calibration Analysis			
Part 3	Determination of the Uncertainty of q for a Single Core Design			
		Case A: No Shared Error Sources in Any Measurements		
			TSM Analysis	
			MCM Analysis	
		Case B: Possible Shared Error Sources in Temperature Measurements		
			TSM Analysis	
			MCM Analysis	
Part 4	Determination of the Uncertainty in Δq for Two Core Designs Tested Sequentially Using the Same Facility and Instrumentation			
			TSM Analysis	
				No shared error sources among the T_1, T_2, and T_3 measurements within a single test (as in Part 3, Case A)
				Shared error sources for the T_1, T_2, and T_3 measurements within a single test (as in Part 3, Case B)
			MCM Analysis	
				No shared error sources among the T_1, T_2, and T_3 measurements within a single test (as in Part 3, Case A)
				Shared error sources for the T_1, T_2, and T_3 measurements within a single test (as in Part 3, Case B)

8-1 PART 1: OVERVIEW

A heat exchanger test facility is used to test heat exchanger cores using a hot air-cooling water configuration, as indicated schematically in Figure 8-1-1.

The test facility where the core is installed contains all required instrumentation; no new instrumentation is necessary for testing different cores. There is one thermocouple probe (T_1) in a well in the water inlet header. There are two spatially separated wells, each with a thermocouple probe (T_2 and T_3), in the water outlet header. A turbine meter is used to determine water volumetric flow rate, Q.

The result of interest is the rate of heat transfer to the cooling water, which is determined for a given set point using the DRE

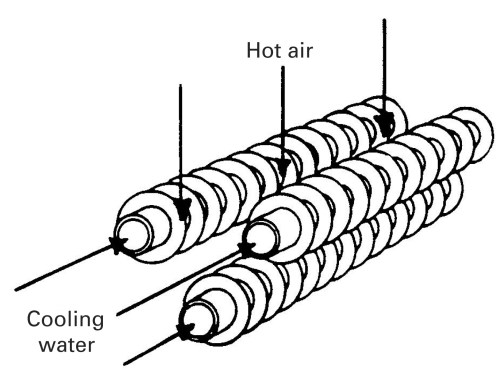
$$q = \rho Qc\left(\frac{T_2 + T_3}{2} - T_1\right)$$

where

c = the constant pressure-specific heat of the water at an average temperature
Q = the volumetric flow rate of the water
q = the rate of heat transfer from the hot air to the cooling water
T_1 = the water temperature in the inlet header as measured by a single probe
T_2, T_3 = temperatures measured by two temperature probes at different positions in a cross-section in the water outlet header
ρ = the water density

Conceptually, this equation assumes a steady state with T_1 corresponding to the averaged water temperature at the inlet plane of the test core and $(T_2 + T_3)/2$ corresponding to the averaged water temperature at the outlet plane of the test core.

Figure 8-1-1 Heat Exchanger Cores Using Hot Air-Cooling Water Configuration



8-1.1 Random Standard Uncertainty for the Result, q

This is a well-established facility with a history that allows use of prior test data to establish a large sample estimate, s_q, of the random standard uncertainty in q for a single core tested multiple times at the same nominal set point and with removal and reinstallation between tests.

8-1.2 Systematic Standard Uncertainties

The systematic standard uncertainties are identified as follows:

for c = a single elemental systematic error source with systematic standard uncertainty, b_c

for Q = a single elemental systematic error source with systematic standard uncertainty, b_Q

for T_1 = two elemental systematic error sources with elemental systematic standard uncertainties, $b_{T1,1}$ and $b_{T1,2}$

for T_2 = two elemental systematic error sources with elemental systematic standard uncertainties, $b_{T2,1}$ and $b_{T2,2}$

for T_3 = two elemental systematic error sources with elemental systematic standard uncertainties, $b_{T3,1}$ and $b_{T3,2}$

for ρ = a single elemental systematic error source with systematic standard uncertainty, b_ρ

This number of elemental error sources is necessary and sufficient for illustrating all of the facets of this example. In specific actual cases, the number of elemental systematic error sources may be greater, but no extension of the approaches illustrated would be necessary other than simply adding more terms to account for the additional sources.

For this example, it is postulated that the thermocouple probes are calibrated in a "constant temperature" water bath containing a rack of test tubes. For a given calibration set point, the probes are placed in individual tubes and the standard against which they are calibrated placed in a separate test tube. In such a case it is stipulated that the first of the elemental systematic error sources for each temperature measurement is from the calibration standard, and the associated elemental standard systematic uncertainties are designated as

$$b_{T1,1} \equiv b_{\text{std},1}$$
$$b_{T2,1} \equiv b_{\text{std},2}$$
$$b_{T3,1} \equiv b_{\text{std},3}$$

For the most general case, the probes could be calibrated individually against different standards and this nomenclature allows for that.

The second of the elemental systematic error sources for each temperature measurement is due to the bath nonuniformity. When the bath is held at a supposed steady state at a calibration set point, it is not at a uniform temperature. The "error" in this case will be the difference between the temperature where the standard is located and the temperature where a probe is located during calibration. While the bath is at a steady-state calibration set point, traversing the standard to different points in the bath and recording the temperature differences from some chosen reference position yields a distribution of temperature differences, and the standard deviation of this distribution is used as the large sample estimate of $b_{T1,2} = b_{T2,2} = b_{T3,2} = b_{\text{bath}}$. Note that although the uncertainties are equal, the errors in each probe due to the elemental source are different if the probes are at different positions in the bath during the calibration. For this reason, it is useful to define

$$b_{T1,2} \equiv b_{\text{bath},1}$$
$$b_{T2,2} \equiv b_{\text{bath},2}$$
$$b_{T3,2} \equiv b_{\text{bath},3}$$

8-2 PART 2: GENERIC CALIBRATION ANALYSIS

Consider a generic thermocouple calibration case. The thermocouple (tc) connected to a data acquisition system (das) consisting of an electronic reference junction, signal conditioning, an analog-to-digital converter, and a digital voltmeter. The tc is exposed to some temperature, T_{true}, that one wishes to measure and the system output is the voltage, E, as shown in illustration (a) of Figure 8-2-1.

Suppose that the thermocouple is used as supplied and is not individually calibrated. In that case, as shown in illustration (b) of Figure 8-2-1, the voltage, E, is used in a generic T-vs-E table for the particular type of thermocouple, and the corresponding temperature, T, is found. This is the temperature that is said to be the "measured" value of T_{true}.

The uncertainty in this value includes contributions from the elemental systematic errors.

$\beta_{a/d}$ = the error from the analog to digital converter

β_{dvm} = the error from the digital voltmeter

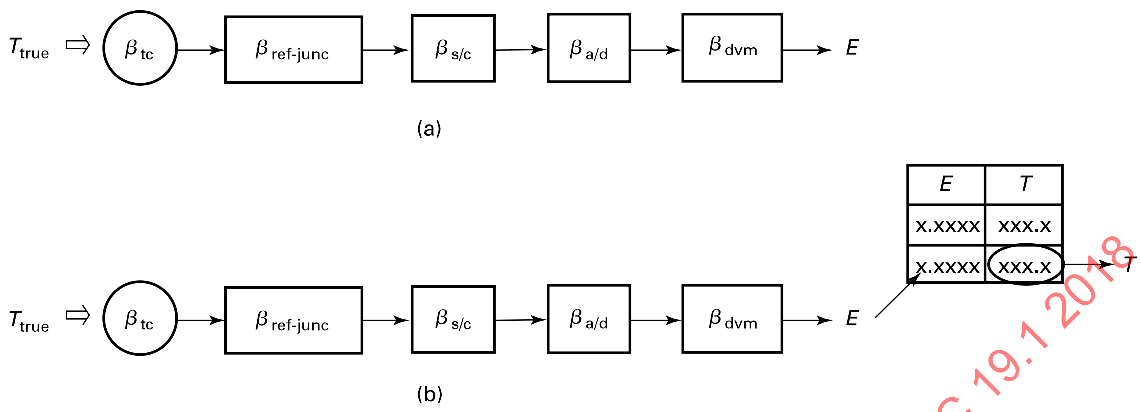
$\beta_{\text{ref-junc}}$ = the error from the electronic reference junction

$\beta_{s/c}$ = the error from the signal conditioner

β_{tc} = the amount this tc differs from the generic tc in the table

Suppose the thermocouple is calibrated as shown in illustration (a) of Figure 8-2-2, where the thermocouple and the temperature standard, T_{std}, are both exposed to the same temperature, T_{true}. The voltage, E, output by the thermocouple system (tc + das) and the temperature, T_{std}, indicated by the standard are entered as a data pair into

Figure 8-2-1 Measurement of a Generic Thermocouple Output



the calibration table, which replaces the generic table used previously.

When the system is used to measure T_{true}, as shown in illustration (b) of Figure 8-2-2, the voltage output, E, is entered in the calibration table and the corresponding temperature, T, retrieved is, in effect, what would be indicated by the standard.

The uncertainty in this resulting temperature now includes the contribution from the systematic error β_{std} (the amount T_{std} differs from T_{true}), which replaces the contributions from the errors β_{tc}, $\beta_{ref\text{-}junc}$, $\beta_{s/c}$, $\beta_{a/d}$, and β_{dvm}.

This shows that some systematic errors can be replaced by (hopefully) smaller ones by careful application of the calibration process.

NOTE: It is critical to identify exactly what is being calibrated.

If the tc and the das that are calibrated together are used in the test, then the error situation is as previously described. However, if the das used in the calibration, das_{cal}, is replaced by another, das_{test}, for the actual test, the systematic error β_{std} from the standard then only replaces the error contribution β_{tc}. The uncertainty in the resulting measured temperature, T, now includes contributions from the error in the standard, β_{std}; those errors from the das used in the calibration ($\beta_{ref\text{-}junc}$, $\beta_{s/c}$, $\beta_{a/d}$, $\beta_{dvm})_{das\text{-}cal}$; and those errors from the das used in the test ($\beta_{ref\text{-}junc}$, $\beta_{s/c}$, $\beta_{a/d}$, $\beta_{dvm})_{das\text{-}test}$.

In the situation stipulated for this comprehensive example, the same das is used during the calibration and the tests, so there are not separate systematic error sources for calibration and test data acquisition systems.

8-3 PART 3: DETERMINATION OF THE UNCERTAINTY IN *q* FOR A SINGLE CORE DESIGN

8-3.1 Case A: No Shared Error Sources in Any Measurements

This case could occur if there were no common calibration of the three thermocouple probes; for instance, if the probes were from different suppliers. For purposes of this case, stipulate that there is only one significant error source for each probe (e.g., the manufacturer's accuracy specification), so the corresponding systematic standard uncertainties are b_{T1}, b_{T2}, and b_{T3}.

8-3.1.1 TSM Analysis. The following illustrates the TSM approach to Case A:

$$q = \rho Qc\left(\frac{T_2 + T_3}{2} - T_1\right)$$

$$b_q^2 = \left(\frac{\partial q}{\partial \rho}\right)^2 b_\rho^2 + \left(\frac{\partial q}{\partial Q}\right)^2 b_Q^2 + \left(\frac{\partial q}{\partial c}\right)^2 b_c^2 + \left(\frac{\partial q}{\partial T1}\right)^2 b_{T1}^2 + \left(\frac{\partial q}{\partial T2}\right)^2 b_{T2}^2 + \left(\frac{\partial q}{\partial T3}\right)^2 b_{T3}^2$$

$$u_q^2 = b_q^2 + s_q^2$$

8-3.1.2 MCM Analysis. Figure 8-3-1 illustrates the Monte Carlo approach to Case A. First, a single "run" of the experiment is constructed with a systematic error drawn from each variable's assumed error distribution, having as its standard deviation the estimated systematic uncertainty for that variable. This is shown in Figure 8-3-1, where uniform distributions are assumed for all systematic error sources. (This assumption is made in this example for the sake of simplicity — it is not necessarily a general recommendation.)

For the i^{th} run, for instance, a "measured" value of each variable is calculated as

Figure 8-2-2 Measurement of a Calibrated Thermocouple Output

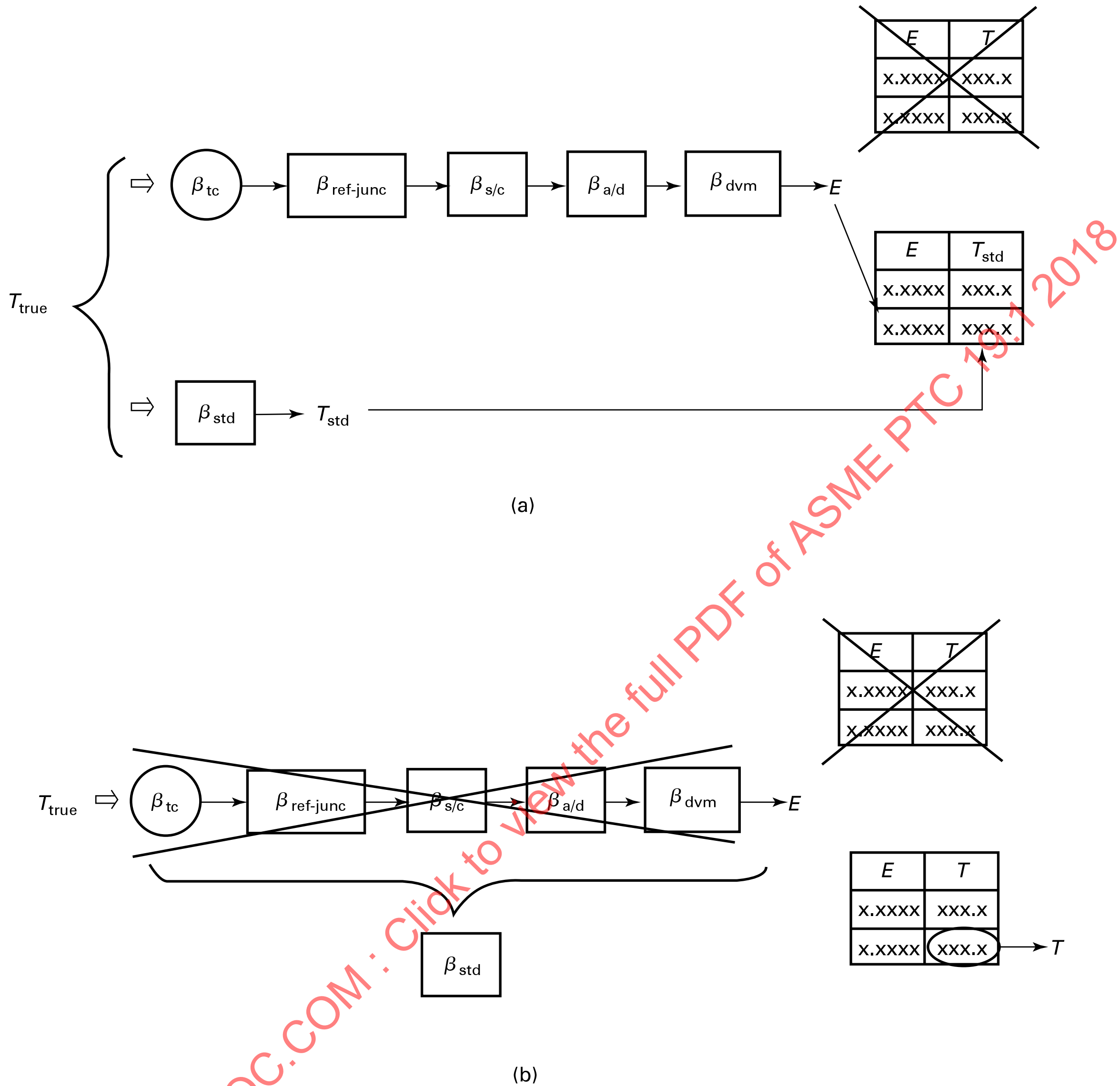
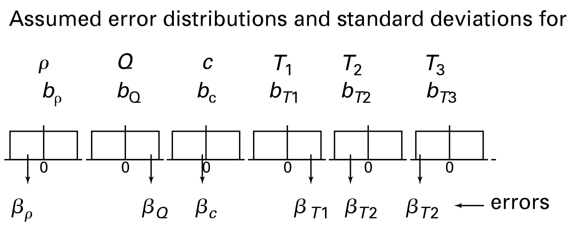


Figure 8-3-1 Monte Carlo Uncertainty Analysis



$$\rho_i = \rho_{\text{true}} + \left(\beta_\rho\right)_i$$
$$Q_i = Q_{\text{true}} + \left(\beta_Q\right)_i$$
$$c_i = c_{\text{true}} + (\beta_c)_i$$
$$T_{1,i} = (T_1)_{\text{true}} + (\beta_{T1})_i$$
$$T_{2,i} = (T_2)_{\text{true}} + (\beta_{T2})_i$$
$$T_{3,i} = (T_3)_{\text{true}} + (\beta_{T3})_i$$

and the value of the result is calculated as

$$q_i = \rho_i Q_i c_i \left(\frac{T_{2,i} + T_{3,i}}{2} - T_{1,i} \right) + \left(\varepsilon_q\right)_i$$

where the random error in the result, $(\varepsilon_q)_i$, is drawn from an assumed error distribution with standard deviation equal to s_q, the random standard uncertainty of q.

When this is repeated M times, a distribution of M values of q is obtained. The standard deviation of this distribution is u_q, the total standard uncertainty in q. A coverage interval can be defined and calculated directly using the M q values, with no assumption necessary about the form of the distribution of the M values.

8-3.2 Case B: Possible Shared Error Sources in Temperature Measurements

This is the situation prescribed in Section 8-1, with temperature measurement elemental error sources from the standard(s) and the bath nonuniformity.

8-3.2.1 TSM Analysis.

$$q = \rho Q c \left(\frac{T_2 + T_3}{2} - T_1 \right)$$

$$b_q^2 = \left(\frac{\partial q}{\partial \rho}\right)^2 b_\rho^2 + \left(\frac{\partial q}{\partial Q}\right)^2 b_Q^2 + \left(\frac{\partial q}{\partial c}\right)^2 b_c^2 + \left(\frac{\partial q}{\partial T1}\right)^2 b_{T1}^2 + \left(\frac{\partial q}{\partial T2}\right)^2 b_{T2}^2 + \left(\frac{\partial q}{\partial T3}\right)^2 b_{T3}^2 + 2\left(\frac{\partial q}{\partial T1}\right)\left(\frac{\partial q}{\partial T2}\right) b_{T1T2} + 2\left(\frac{\partial q}{\partial T1}\right)\left(\frac{\partial q}{\partial T3}\right) b_{T1T3} + 2\left(\frac{\partial q}{\partial T2}\right)\left(\frac{\partial q}{\partial T3}\right) b_{T2T3}$$

$$b_{T1}^2 = b_{T\text{std},1}^2 + b_{T\text{bath},1}^2$$
$$b_{T2}^2 = b_{T\text{std},2}^2 + b_{T\text{bath},2}^2$$
$$b_{T3}^2 = b_{\text{std},3}^2 + b_{T\text{bath},3}^2$$

Consider if the three thermocouple probes were calibrated against different standards and were all in different positions in the bath during calibration. This unlikely case is worth considering in order to see the logical progression of the following analyses. Then there would be no shared error sources and $b_{T1T2} = b_{T1T3} = b_{T2T3} = 0$.

If the three thermocouple probes were calibrated against the same standard but were all in different positions in the bath during calibration, then

$$b_{T1T2} = b_{T\text{std},1} b_{T\text{std},2}$$
$$b_{T1T3} = b_{T\text{std},1} b_{T\text{std},3}$$
$$b_{T2T3} = b_{T\text{std},2} b_{T\text{std},3}$$

If the three thermocouple probes were calibrated against the same standard and were all in the same position in the bath (but at a different location than the standard during calibration), then

$$b_{T1T2} = b_{T\text{std},1} b_{T\text{std},2} + b_{T\text{bath},1} b_{T\text{bath},2}$$
$$b_{T1T3} = b_{T\text{std},1} b_{T\text{std},3} + b_{T\text{bath},1} b_{T\text{bath},2}$$
$$b_{T2T3} = b_{T\text{std},2} b_{T\text{std},3} + b_{T\text{bath},2} b_{T\text{bath},3}$$

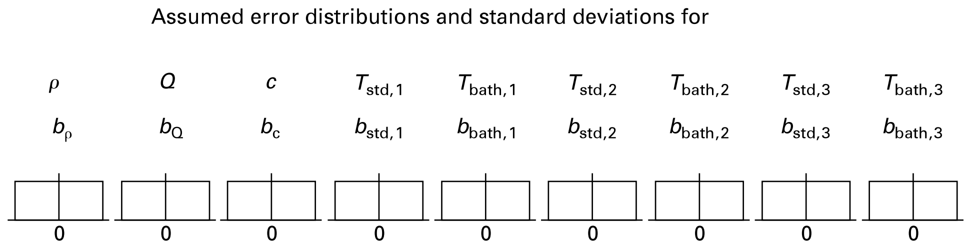
The combined standard uncertainty in q is then given by

$$u_q^2 = b_q^2 + s_q^2$$

The derivatives with respect to ρ, Q, and c are functions of the measured temperatures, but the derivatives with respect to the temperatures are not functions of the temperatures themselves, and the expression given herein can be algebraically simplified. The derivatives with respect to the temperatures are

$$\frac{\partial q}{\partial T_1} = -\rho Q c$$
$$\frac{\partial q}{\partial T_2} = \frac{\partial q}{\partial T_3} = \frac{1}{2}\rho Q c$$

Figure 8-3.2-1 Uniform Distributions for Elemental Systematic Error Sources



Substituting for these derivatives, the equation for b_q can now be written as

$$\begin{aligned} b_q^2 &= \left(\frac{\partial q}{\partial \rho}\right)^2 b_\rho^2 + \left(\frac{\partial q}{\partial Q}\right)^2 b_Q^2 + \left(\frac{\partial q}{\partial c}\right)^2 b_c^2 + (\rho Q c)^2 b_{T1}^2 \\ &\quad + \frac{1}{4}(\rho Q c)^2 b_{T2}^2 + \frac{1}{4}(\rho Q c)^2 b_{T3}^2 \\ &\quad + 2(-\rho Q c)\left(\frac{1}{2}\rho Q c\right) b_{T1T2} + 2(-\rho Q c) \\ &\quad \left(\frac{1}{2}\rho Q c\right) b_{T1T3} \\ &\quad + 2\left(\frac{1}{2}\rho Q c\right)\left(\frac{1}{2}\rho Q c\right) b_{T2T3} \end{aligned}$$

or

$$\begin{aligned} b_q^2 &= \left(\frac{\partial q}{\partial \rho}\right)^2 b_\rho^2 + \left(\frac{\partial q}{\partial Q}\right)^2 b_Q^2 + \left(\frac{\partial q}{\partial c}\right)^2 b_c^2 + (\rho Q c)^2 b_{T1}^2 \\ &\quad + \frac{1}{4}(\rho Q c)^2 b_{T2}^2 + \frac{1}{4}(\rho Q c)^2 b_{T3}^2 \\ &\quad - (\rho Q c)^2 b_{T1T2} - (\rho Q c)^2 b_{T1T3} \\ &\quad + \frac{1}{2}(\rho Q c)^2 b_{T2T3} \end{aligned}$$

Thus two of the correlation terms are negative and one is positive. This indicates the possibility of decreasing b_q by proper choice(s) of calibration, forcing correlation of some error sources but not others.

8-3.2.2 MCM Analysis. As in the Monte Carlo approach shown in Case A, first a single "run" of the experiment is constructed with a systematic error β drawn for each elemental source from an assumed error distribution. This has as its standard deviation the estimated systematic standard uncertainty for that elemental source. Figure 8-3.2-1 illustrates this with uniform distributions assumed for all elemental systematic error sources.

Consider if the three thermocouple probes were calibrated against different standards and were all in different positions in the bath during calibration. This is an unlikely case, but worth considering in order to see the logical progression of the analyses. Then there would be no shared error sources and single errors β would be drawn from each of the distributions in Figure 8-3.2-1. This gives

$$\begin{aligned} \rho_i &= \rho_{\text{true}} + \left(\beta_\rho\right)_i \\ Q_i &= Q_{\text{true}} + \left(\beta_Q\right)_i \\ c_i &= c_{\text{true}} + (\beta_c)_i \\ T_{1,i} &= (T_1)_{\text{true}} + \left(\beta_{\text{std},1}\right)_i + \left(\beta_{\text{bath},1}\right)_i \\ T_{2,i} &= (T_2)_{\text{true}} + \left(\beta_{\text{std},2}\right)_i + \left(\beta_{\text{bath},2}\right)_i \\ T_{3,1} &= (T_3)_{\text{true}} + \left(\beta_{\text{std},3}\right)_i + \left(\beta_{\text{bath},3}\right)_i \end{aligned}$$

and the value of the result is calculated as

$$q_i = \rho_i Q_i c_i \left(\frac{T_{2,i} + T_{3,i}}{2} - T_{1,i}\right) + (\varepsilon_q)_i$$

where the random error in the result, $(\varepsilon_q)_i$, is drawn from an assumed error distribution with standard deviation equal to s_q, the random standard uncertainty of q.

If the three thermocouple probes were calibrated against the same standard but were all in different positions in the bath during calibration, then the error from elemental source 1 (the standard) would be exactly the same for each of the three temperature measurements during Monte Carlo iteration i. This is modeled by drawing a single error β_{std} during iteration i from the $T_{\text{std},1}$ error distribution and setting

$$\left(\beta_{\text{std},1}\right)_i = \left(\beta_{\text{std},2}\right)_i = \left(\beta_{\text{std},3}\right)_i \equiv (\beta_{\text{std}})_i$$

so that

$$\begin{aligned} T_{1,i} &= (T_1)_{\text{true}} + (\beta_{\text{std}})_i + \left(\beta_{\text{bath},1}\right)_i \\ T_{2,i} &= (T_2)_{\text{true}} + (\beta_{\text{std}})_i + \left(\beta_{\text{bath},2}\right)_i \\ T_{3,1} &= (T_3)_{\text{true}} + (\beta_{\text{std}})_i + \left(\beta_{\text{bath},3}\right)_i \end{aligned}$$

and the value of the result is again calculated as

$$q_i = \rho_i Q_i c_i \left(\frac{T_{2,i} + T_{3,i}}{2} - T_{1,i} \right) + (\varepsilon_q)_i$$

where the random error in the result, $(\varepsilon_q)_i$, is drawn from an assumed error distribution with standard deviation equal to s_q, the random standard uncertainty of q.

If the three thermocouple probes were calibrated against the same standard and were all at the same position in the bath (but not necessarily the same position as the standard) during calibration, then the error from elemental source 1 (the standard) would be exactly the same for each of the three temperature measurements during Monte Carlo iteration i, and also the error from elemental source 2 (the bath nonuniformity) would be exactly the same for each of the three temperature measurements during Monte Carlo iteration i. This is modeled by drawing a single error β_{std} during iteration i from the $T_{std,1}$ error distribution and setting

$$\left(\beta_{std,1}\right)_i = \left(\beta_{std,2}\right)_i = \left(\beta_{std,3}\right)_i \equiv (\beta_{std})_i$$

and drawing a single error β_{bath} during iteration i from the $T_{bath,1}$ error distribution and setting

$$\left(\beta_{bath,1}\right)_i = \left(\beta_{bath,2}\right)_i = \left(\beta_{bath,3}\right)_i \equiv (\beta_{bath})_i$$

so that

$$\begin{aligned} T_{1,i} &= (T_1)_{true} + (\beta_{std})_i + (\beta_{bath})_i \\ T_{2,i} &= (T_2)_{true} + (\beta_{std})_i + (\beta_{bath})_i \\ T_{3,i} &= (T_3)_{true} + (\beta_{std})_i + (\beta_{bath})_i \end{aligned}$$

and the value of the result is again calculated as

$$q_i = \rho_i Q_i c_i \left(\frac{T_{2,i} + T_{3,i}}{2} - T_{1,i} \right) + (\varepsilon_q)_i$$

where the random error in the result, $(\varepsilon_q)_i$, is drawn from an assumed error distribution with standard deviation equal to s_q, the random standard uncertainty of q.

When this is repeated M times, a distribution of M values of q is obtained. The standard deviation of this distribution is u_q, the total standard uncertainty in q. A coverage interval can be defined and calculated directly using the M q values, with no assumption necessary about the form of the distribution of the M values.

8-4 PART 4: DETERMINATION OF THE UNCERTAINTY IN Δ*q* FOR TWO CORE DESIGNS TESTED SEQUENTIALLY USING THE SAME FACILITY AND INSTRUMENTATION

Labeling the first design as f and the second design as g, the DRE for the difference in the rates of heat transfer determined for the two designs is

$$\Delta q = q_f - q_g$$

or

$$\begin{aligned} \Delta q = {} & \rho_f Q_f c_f \left(\frac{T_{2,f} + T_{3,f}}{2} - T_{1,f} \right) \\ & - \rho_g Q_g c_g \left(\frac{T_{2,g} + T_{3,g}}{2} - T_{1,g} \right) \end{aligned}$$

8-4.1 Random Uncertainty for the Result Δ*q*

For this example, assume that s_q is a valid estimate for both tests f and g, so that the TSM propagation equation is used to estimate the random standard uncertainty of Δq:

$$s_{\Delta q}^2 = \left(\frac{\partial \Delta q}{\partial q_f} \right)^2 s_q^2 + \left(\frac{\partial \Delta q}{\partial q_g} \right)^2 s_q^2$$

Since the partial derivatives are +1 and -1, this gives

$$s_{\Delta q} = \sqrt{2}\, s_q$$

8-4.2 TSM Analysis: Systematic Standard Uncertainty for the Result Δ*q*

8-4.2.1 No shared error sources among the T_1, T_2, and T_3 measurements within a single test (as in Case A). If there are no shared error sources among the T_1, T_2, and T_3 measurements within a single test (as in Case A in para. 8-3.1), the TSM gives

$$
\begin{aligned}
b_{\Delta q}^2 = {} & \left(\frac{\partial \Delta q}{\partial \rho}\right)_f^2 b_{\rho_f}^2 + \left(\frac{\partial \Delta q}{\partial \rho}\right)_g^2 b_{\rho_g}^2 \\
& +2\left(\frac{\partial \Delta q}{\partial \rho}\right)_f \left(\frac{\partial \Delta q}{\partial \rho}\right)_g b_{\rho_f \rho_g} \\
& +\left(\frac{\partial \Delta q}{\partial Q}\right)_f^2 b_{Q_f}^2 + \left(\frac{\partial \Delta q}{\partial Q}\right)_g^2 b_{Q_g}^2 \\
& +2\left(\frac{\partial \Delta q}{\partial Q}\right)_f \left(\frac{\partial \Delta q}{\partial Q}\right)_g b_{Q_f Q_g} \\
& +\left(\frac{\partial \Delta q}{\partial c}\right)_f^2 b_{c_f}^2 + \left(\frac{\partial \Delta q}{\partial c}\right)_g^2 b_{c_g}^2 \\
& +2\left(\frac{\partial \Delta q}{\partial c}\right)_f \left(\frac{\partial \Delta q}{\partial c}\right)_g b_{c_f c_g} \\
& +\left(\frac{\partial \Delta q}{\partial T1}\right)_f^2 b_{T1,f}^2 + \left(\frac{\partial \Delta q}{\partial T1}\right)_g^2 b_{T1,g}^2 \\
& +2\left(\frac{\partial \Delta q}{\partial T1}\right)_f \left(\frac{\partial \Delta q}{\partial T1}\right)_g b_{T1_f T1_g} \\
& +\left(\frac{\partial \Delta q}{\partial T2}\right)_f^2 b_{T2,f}^2 + \left(\frac{\partial \Delta q}{\partial T2}\right)_g^2 b_{T2,g}^2 \\
& +2\left(\frac{\partial \Delta q}{\partial T2}\right)_f \left(\frac{\partial \Delta q}{\partial T2}\right)_g b_{T2_f T2_g} \\
& +\left(\frac{\partial \Delta q}{\partial T3}\right)_f^2 b_{T3,f}^2 + \left(\frac{\partial \Delta q}{\partial T3}\right)_g^2 b_{T3,g}^2 \\
& +2\left(\frac{\partial \Delta q}{\partial T3}\right)_f \left(\frac{\partial \Delta q}{\partial T3}\right)_g b_{T3_f T3_g}
\end{aligned}
$$

Since the same instrumentation is used in tests f and g, each measured variable in f will share error source(s) with the corresponding measured variable in g.

The derivatives with respect to ρ, Q, and c are functions of the measured temperatures, but the derivatives with respect to the temperatures are not functions of the temperatures themselves, and the expression herein can be algebraically simplified. The derivatives with respect to the temperatures are

$$
\begin{aligned}
\frac{\partial \Delta q}{\partial T_{1,f}} &= -(\rho Q c)_f \\
\frac{\partial \Delta q}{\partial T_{2,f}} &= \frac{\partial \Delta q}{\partial T_{3,f}} = \frac{(\rho Q c)_f}{2} \\
\frac{\partial \Delta q}{\partial T_{1,g}} &= (\rho Q c)_g \\
\frac{\partial \Delta q}{\partial T_{2,g}} &= \frac{\partial \Delta q}{\partial T_{3,g}} = -\frac{(\rho Q c)_g}{2}
\end{aligned}
$$

Substituting for these derivatives, the TSM expression for $b_{\Delta q}$ can now be written as

$$
\begin{aligned}
b_{\Delta q}^2 = {} & \left(\frac{\partial \Delta q}{\partial \rho}\right)_f^2 b_{\rho_f}^2 + \left(\frac{\partial \Delta q}{\partial \rho}\right)_g^2 b_{\rho_g}^2 \\
& +2\left(\frac{\partial \Delta q}{\partial \rho}\right)_f \left(\frac{\partial \Delta q}{\partial \rho}\right)_g b_{\rho_f \rho_g} \\
& +\left(\frac{\partial \Delta q}{\partial Q}\right)_f^2 b_{Q_f}^2 + \left(\frac{\partial \Delta q}{\partial Q}\right)_g^2 b_{Q_g}^2 \\
& +2\left(\frac{\partial \Delta q}{\partial Q}\right)_f \left(\frac{\partial \Delta q}{\partial Q}\right)_g b_{Q_f Q_g} \\
& +\left(\frac{\partial \Delta q}{\partial c}\right)_f^2 b_{c_f}^2 + \left(\frac{\partial \Delta q}{\partial c}\right)_g^2 b_{c_g}^2 \\
& +2\left(\frac{\partial \Delta q}{\partial c}\right)_f \left(\frac{\partial \Delta q}{\partial c}\right)_g b_{c_f c_g} \\
& +(\rho Q c)_f^2 b_{T1,f}^2 + (\rho Q c)_g^2 b_{T1,g}^2 \\
& -2(\rho Q c)_f (\rho Q c)_g b_{T1_f T1_g} \\
& +\frac{1}{4}(\rho Q c)_f^2 b_{T2,f}^2 + \frac{1}{4}(\rho Q c)_g^2 b_{T2,g}^2 \\
& -\frac{1}{2}(\rho Q c)_f (\rho Q c)_g b_{T2_f T2_g} + \frac{1}{4}(\rho Q c)_f^2 b_{T3,f}^2 \\
& +\frac{1}{4}(\rho Q c)_g^2 b_{T3,g}^2 \\
& -\frac{1}{2}(\rho Q c)_f (\rho Q c)_g b_{T3_f T3_g}
\end{aligned}
$$

Each measured temperature in test g will have identical error sources to that same temperature measured in test f, so that

$$
\begin{aligned}
b_{T1,f} &= b_{T1,g} = \sqrt{b_{\text{std},1}{}^2 + b_{\text{bath},1}{}^2} \equiv b_{T1} \\
b_{T2,f} &= b_{T2,g} = \sqrt{b_{\text{std},2}{}^2 + b_{\text{bath},2}{}^2} \equiv b_{T2} \\
b_{T3,f} &= b_{T3,g} = \sqrt{b_{\text{std},3}{}^2 + b_{\text{bath},3}{}^2} \equiv b_{T3}
\end{aligned}
$$

and

$$
\begin{aligned}
b_{T1_f T1_g} &= b_{\text{std},1} b_{\text{std},1} + b_{\text{bath},1} b_{\text{bath},1} = b_{\text{std},1}^2 + b_{\text{bath},1}^2 \\
&\equiv b_{T1}^2 \\
b_{T2_f T2_g} &= b_{\text{std},2} b_{\text{std},2} + b_{\text{bath},2} b_{\text{bath},2} = b_{\text{std},2}^2 + b_{\text{bath},2}^2 \\
&\equiv b_{T2}^2 \\
b_{T3_f T3_g} &= b_{\text{std},3} b_{\text{std},3} + b_{\text{bath},3} b_{\text{bath},3} = b_{\text{std},3}^2 + b_{\text{bath},3}^2 \\
&\equiv b_{T3}^2
\end{aligned}
$$

For the situation in which the two tests are run at identical set points,

$$
(\rho Q c)_f \cong (\rho Q c)_g \equiv (\rho Q c)
$$

and the final nine terms in the equation for $b_{\Delta q}$ become

$$
\begin{aligned}
&+(\rho Qc)_f^2 b_{T1,f}^2 \\
&+(\rho Qc)_g^2 b_{T1,g}^2 \\
&-2(\rho Qc)_f(\rho Qc)_g b_{T1_f T1_g} \\
&+\frac{1}{4}(\rho Qc)_f^2 b_{T2,f}^2 \\
&+\frac{1}{4}(\rho Qc)_g^2 b_{T2,g}^2 \\
&-\frac{1}{2}(\rho Qc)_f(\rho Qc)_g b_{T2_f T2_g} \\
&+\frac{1}{4}(\rho Qc)_f^2 b_{T3,f}^2 \\
&+\frac{1}{4}(\rho Qc)_g^2 b_{T3,g}^2 \\
&-\frac{1}{2}(\rho Qc)_f(\rho Qc)_g b_{T3_f T3_g} \\
&= +\ (\rho Qc)^2 b_{T1}^2 + (\rho Qc)^2 b_{T1}^2 \\
&-2(\rho Qc)^2 b_{T1}^2 \\
&+\frac{1}{4}(\rho Qc)^2 b_{T2}^2 + \frac{1}{4}(\rho Qc)^2 b_{T2}^2 \\
&-\frac{1}{2}(\rho Qc)^2 b_{T2}^2 + \frac{1}{4}(\rho Qc)^2 b_{T3}^2 \\
&+\frac{1}{4}(\rho Qc)^2 b_{T3}^2 \\
&-\frac{1}{2}(\rho Qc)^2 b_{T3}^2 \equiv 0
\end{aligned}
$$

Therefore, for the stated conditions, the effects of the temperature elemental systematic error sources from the standard and the bath nonuniformity totally cancel out and the equation for $b_{\Delta q}$ becomes

$$
\begin{aligned}
b_{\Delta q}^2 = &\left(\frac{\partial \Delta q}{\partial \rho}\right)_f^2 b_{\rho_f}^2 + \left(\frac{\partial \Delta q}{\partial \rho}\right)_g^2 b_{\rho_g}^2 \\
&+2\left(\frac{\partial \Delta q}{\partial \rho}\right)_f\left(\frac{\partial \Delta q}{\partial \rho}\right)_g b_{\rho_f \rho_g} \\
&+\left(\frac{\partial \Delta q}{\partial Q}\right)_f^2 b_{Q_f}^2 + \left(\frac{\partial \Delta q}{\partial Q}\right)_g^2 b_{Q_g}^2 \\
&+2\left(\frac{\partial \Delta q}{\partial Q}\right)_f\left(\frac{\partial \Delta q}{\partial Q}\right)_g b_{Q_f Q_g} \\
&+\left(\frac{\partial \Delta q}{\partial c}\right)_f^2 b_{c_f}^2 + \left(\frac{\partial \Delta q}{\partial c}\right)_g^2 b_{c_g}^2 \\
&+2\left(\frac{\partial \Delta q}{\partial c}\right)_f\left(\frac{\partial \Delta q}{\partial c}\right)_g b_{c_f c_g}
\end{aligned}
$$

8-4.2.2 Shared error sources for the T_1, T_2, and T_3 measurements within a single test (as in Case B). If there are shared error sources for the T_1, T_2, and T_3 measurements within a single test (as in Case B in para. 8-3.2), the TSM gives

$$
\begin{aligned}
b_{\Delta q}^2 = &\left(\frac{\partial \Delta q}{\partial \rho}\right)_f^2 b_{\rho_f}^2 + \left(\frac{\partial \Delta q}{\partial \rho}\right)_g^2 b_{\rho_g}^2 \\
&+2\left(\frac{\partial \Delta q}{\partial \rho}\right)_f\left(\frac{\partial \Delta q}{\partial \rho}\right)_g b_{\rho_f \rho_g} \\
&+\left(\frac{\partial \Delta q}{\partial Q}\right)_f^2 b_{Q_f}^2 + \left(\frac{\partial \Delta q}{\partial Q}\right)_g^2 b_{Q_g}^2 \\
&+2\left(\frac{\partial \Delta q}{\partial Q}\right)_f\left(\frac{\partial \Delta q}{\partial Q}\right)_g b_{Q_f Q_g} \\
&+\left(\frac{\partial \Delta q}{\partial c}\right)_f^2 b_{c_f}^2 + \left(\frac{\partial \Delta q}{\partial c}\right)_g^2 b_{c_g}^2 \\
&+2\left(\frac{\partial \Delta q}{\partial c}\right)_f\left(\frac{\partial \Delta q}{\partial c}\right)_g b_{c_f c_g} \\
&+\left(\frac{\partial \Delta q}{\partial T1}\right)_f^2 b_{T1,f}^2 + \left(\frac{\partial \Delta q}{\partial T1}\right)_g^2 b_{T1,g}^2 \\
&+2\left(\frac{\partial \Delta q}{\partial T1}\right)_f\left(\frac{\partial \Delta q}{\partial T1}\right)_g b_{T1_f T1_g} \\
&+\left(\frac{\partial \Delta q}{\partial T2}\right)_f^2 b_{T2,f}^2 + \left(\frac{\partial \Delta q}{\partial T2}\right)_g^2 b_{T2,g}^2 \\
&+2\left(\frac{\partial \Delta q}{\partial T2}\right)_f\left(\frac{\partial \Delta q}{\partial T2}\right)_g b_{T2_f T2_g} \\
&+\left(\frac{\partial \Delta q}{\partial T3}\right)_f^2 b_{T3,f}^2 + \left(\frac{\partial \Delta q}{\partial T3}\right)_g^2 b_{T3,g}^2 \\
&+2\left(\frac{\partial \Delta q}{\partial T3}\right)_f\left(\frac{\partial \Delta q}{\partial T3}\right)_g b_{T3_f T3_g} \\
&+2\left(\frac{\partial \Delta q}{\partial T1}\right)_f\left(\frac{\partial \Delta q}{\partial T2}\right)_f b_{T1_f T2_f} \\
&+2\left(\frac{\partial \Delta q}{\partial T1}\right)_f\left(\frac{\partial \Delta q}{\partial T3}\right)_f b_{T1_f T3_f} \\
&+2\left(\frac{\partial \Delta q}{\partial T1}\right)_f\left(\frac{\partial \Delta q}{\partial T2}\right)_g b_{T1_f T2_g} \\
&+2\left(\frac{\partial \Delta q}{\partial T1}\right)_f\left(\frac{\partial \Delta q}{\partial T3}\right)_g b_{T1_f T3_g} \\
&+2\left(\frac{\partial \Delta q}{\partial T2}\right)_f\left(\frac{\partial \Delta q}{\partial T3}\right)_f b_{T2_f T3_f} \\
&+2\left(\frac{\partial \Delta q}{\partial T2}\right)_f\left(\frac{\partial \Delta q}{\partial T1}\right)_g b_{T2_f T1_g} \\
&+2\left(\frac{\partial \Delta q}{\partial T2}\right)_f\left(\frac{\partial \Delta q}{\partial T3}\right)_g b_{T2_f T3_g} \\
&+2\left(\frac{\partial \Delta q}{\partial T3}\right)_f\left(\frac{\partial \Delta q}{\partial T1}\right)_g b_{T3_f T1_g} \\
&+2\left(\frac{\partial \Delta q}{\partial T3}\right)_f\left(\frac{\partial \Delta q}{\partial T2}\right)_g b_{T3_f T2_g}
\end{aligned}
$$

$$+2\left(\frac{\partial \Delta q}{\partial T1}\right)_g\left(\frac{\partial \Delta q}{\partial T2}\right)_g b_{T1_gT2_g}$$

$$+2\left(\frac{\partial \Delta q}{\partial T1}\right)_g\left(\frac{\partial \Delta q}{\partial T3}\right)_g b_{T1_gT3_g}$$

$$+2\left(\frac{\partial \Delta q}{\partial T2}\right)_g\left(\frac{\partial \Delta q}{\partial T3}\right)_g b_{T2_gT3_g}$$

If all three temperature probes are calibrated against the same standard so that

$$b_{\text{std},1} = b_{\text{std},2} = b_{\text{std},3} \equiv b_{\text{std}}$$

and all three temperature probes are at the same position in the bath during calibration (so that the error due to bath nonuniformity is identical for all probes)

$$b_{\text{bath},1} = b_{\text{bath},2} = b_{\text{bath},3} \equiv b_{\text{bath}}$$

then each of the $(b_{Ti})^2$ and b_{TiTj} factors is equal to

$$b_T^2 = b_{\text{std}}^2 + b_{\text{bath}}^2$$

for the situation in which

$$(\rho Qc)_f \cong (\rho Qc)_g \equiv \rho Qc$$

Once again, the effects of the temperature elemental systematic error sources from the standard and the bath nonuniformity totally cancel out and the equation for $b_{\Delta q}$ becomes

$$\begin{aligned}
b_{\Delta q}^2 = &\left(\frac{\partial \Delta q}{\partial \rho}\right)_f^2 b_{\rho_f}^2 + \left(\frac{\partial \Delta q}{\partial \rho}\right)_g^2 b_{\rho_g}^2 \\
&+2\left(\frac{\partial \Delta q}{\partial \rho}\right)_f\left(\frac{\partial \Delta q}{\partial \rho}\right)_g b_{\rho_f\rho_g} \\
&+\left(\frac{\partial \Delta q}{\partial Q}\right)_f^2 b_{Q_f}^2 + \left(\frac{\partial \Delta q}{\partial Q}\right)_g^2 b_{Q_g}^2 \\
&+2\left(\frac{\partial \Delta q}{\partial Q}\right)_f\left(\frac{\partial \Delta q}{\partial Q}\right)_g b_{Q_fQ_g} + \left(\frac{\partial \Delta q}{\partial c}\right)_f^2 b_{c_f}^2 \\
&+\left(\frac{\partial \Delta q}{\partial c}\right)_g^2 b_{c_g}^2 \\
&+2\left(\frac{\partial \Delta q}{\partial c}\right)_f\left(\frac{\partial \Delta q}{\partial c}\right)_g b_{c_fc_g}
\end{aligned}$$

8-4.3 MCM Analysis.

As in the Monte Carlo approach in para. 8-3.2.2, first a single "run," i, of the experimental determination of Δq is constructed. Since for this example it is stipulated that tests f and g are run "back-to-back" using identical instrumentation, the same errors will affect the measured variables in test f as affect the measured variables in test g. This is modeled for run i with a single systematic error, β (the value of which is used in both test f and test g), drawn for each elemental source from an assumed error distribution having as its standard deviation the estimated systematic standard uncertainty for that elemental source. This is shown in Figure 8-3.2-1, assuming uniform distributions for all elemental systematic error sources.

8-4.3.1 No shared error sources among the T_1, T_2, and T_3 measurements within a single test (as in Case A). If there are no shared error sources among the T_1, T_2, and T_3 measurements within a single test (as in Case A in para. 8-3.1), the MCM analysis gives

$$\begin{aligned}
\rho_{i,f} &= \rho_{\text{true},f} + \left(\beta_\rho\right)_i \\
\rho_{i,g} &= \rho_{\text{true},g} + \left(\beta_\rho\right)_i \\
Q_{i,f} &= Q_{\text{true},f} + \left(\beta_Q\right)_i \\
Q_{i,g} &= Q_{\text{true},g} + \left(\beta_Q\right)_i \\
c_{i,f} &= c_{\text{true},f} + (\beta_c)_i \\
c_{i,g} &= c_{\text{true},g} + (\beta_c)_i \\
T_{1,i,f} &= (T_1)_{\text{true},f} + \left(\beta_{\text{std},1}\right)_i + \left(\beta_{\text{bath},1}\right)_i \\
T_{1,i,g} &= (T_1)_{\text{true},g} + \left(\beta_{\text{std},1}\right)_i + \left(\beta_{\text{bath},1}\right)_i \\
T_{2,i,f} &= (T_2)_{\text{true},f} + \left(\beta_{\text{std},2}\right)_i + \left(\beta_{\text{bath},2}\right)_i \\
T_{2,i,g} &= (T_2)_{\text{true},g} + \left(\beta_{\text{std},2}\right)_i + \left(\beta_{\text{bath},2}\right)_i \\
T_{3,i,f} &= (T_3)_{\text{true},f} + \left(\beta_{\text{std},3}\right)_i + \left(\beta_{\text{bath},3}\right)_i \\
T_{3,i,g} &= (T_3)_{\text{true},g} + \left(\beta_{\text{std},3}\right)_i + \left(\beta_{\text{bath},3}\right)_i
\end{aligned}$$

and the value of the result is calculated as

$$\begin{aligned}
\Delta q_i = &\rho_{i,f}Q_{i,f}c_{i,f}\left(\frac{T_{2,i,f} + T_{3,i,f}}{2} - T_{1,if}\right) \\
&- \rho_{i,g}Q_{i,g}c_{i,g}\left(\frac{T_{2,i,g} + T_{3,i,g}}{2} - T_{1,i,g}\right) + (\varepsilon_{\Delta q})_i
\end{aligned}$$

where the random error in the result, $(\varepsilon_{\Delta q})_i$, is drawn from an assumed error distribution with standard deviation equal to $s_{\Delta q}$, the random standard uncertainty of Δq.

When this is repeated M times, a distribution of M values of Δq is obtained. The standard deviation of this distribution is $u_{\Delta q}$, the total standard uncertainty in Δq. A coverage interval can be defined and calculated directly using the M Δq values, with no assumption necessary about the form of the distribution of the M values.

8-4.3.2 Shared error sources for the T_1, T_2, and T_3 measurements within a single test (as in Case B). If there are shared error sources for the T_1, T_2, and T_3 measurements within a single test (as in Case B in para. 8-3.2), it is very simple to take these additional effects into account in the MCM. If, for instance, temperature probes 2 and 3 are calibrated against the same standard but temperature probe 1 is calibrated against another standard, those elemental errors that are shared are all set equal within MCM run i:

$$\left(\beta_{\text{std},2}\right)_i = \left(\beta_{\text{std},3}\right)_i$$

and the equations for calculating $T_{3,i,f}$ and $T_{3,i,g}$ are modified to become

$$\begin{aligned} T_{3,i,f} &= (T_3)_{\text{true},f} + \left(\beta_{\text{std},2}\right)_i + \left(\beta_{\text{bath},3}\right)_i \\ T_{3,i,g} &= (T_3)_{\text{true},g} + \left(\beta_{\text{std},2}\right)_i + \left(\beta_{\text{bath},3}\right)_i \end{aligned}$$

If the same standard is used for all three probes, then in all of the temperature equations the error from the standard for run i will be exactly the same:

$$\left(\beta_{\text{std},1}\right)_i = \left(\beta_{\text{std},2}\right)_i = \left(\beta_{\text{std},3}\right)_i \equiv (\beta_{\text{std}})_i$$

Likewise, if all three probes are at the same position in the bath during calibration, then in all of the temperature equations the error from the bath nonuniformity for run i will be exactly the same

$$\left(\beta_{\text{bath},1}\right)_i = \left(\beta_{\text{bath},2}\right)_i = \left(\beta_{\text{bath},3}\right)_i \equiv (\beta_{\text{bath}})_i$$