RAPPORT TECHNIQUE

TECHNICAL REPORT

CEI IEC TR 61282-3

First edition 2002-08

Fibre optic communication system design guides –

Part 3: Calculation of polarization mode dispersion

Guides de conception des systèmes de communication à fibres optiques –

Partie 3: Calcul de la dispersion en mode de polarisation

Numéro de référence
Reference number
CEI/IEC/TR 61282-3:2002

Publication numbering

As from 1 January 1997 all IEC publications are issued with a designation in the 60000 series. For example, IEC 34-1 is now referred to as IEC 60034-1.

Consolidated editions

The IEC is now publishing consolidated versions of its publications. For example, edition numbers 1.0, 1.1 and 1.2 refer, respectively, to the base publication, the base publication incorporating amendment 1 and the base publication incorporating amendments 1 and 2.

Further information on IEC publications

The technical content of IEC publications is kept under constant review by the IEC, thus ensuring that the content reflects current technology. Information relating to this publication, including its validity, is available in the IEC Catalogue of publications (see below) in addition to new editions, amendments and corrigenda. Information on the subjects under consideration and work in progress undertaken by the technical committee which has prepared this publication, as well as the list of publications issued, is also available from the following:

- **IEC Web Site (www.iec.ch)**

- **Catalogue of IEC publications**

  The on-line catalogue on the IEC web site (www.iec.ch/catlg-e.htm) enables you to search by a variety of criteria including text searches, technical committees and date of publication. On-line information is also available on recently issued publications, withdrawn and replaced publications, as well as corrigenda.

- **IEC Just Published**

  This summary of recently issued publications (www.iec.ch/JP.htm) is also available by email. Please contact the Customer Service Centre (see below) for further information.

- **Customer Service Centre**

  If you have any questions regarding this publication or need further assistance, please contact the Customer Service Centre:

  Email: custserv@iec.ch
  Tel: +41 22 919 02 11
  Fax: +41 22 919 03 00

RAPPORT TECHNIQUE

TECHNICAL REPORT

CEI IEC TR 61282-3

First edition
2002-08

Fibre optic communication system design guides –

Part 3: Calculation of polarization mode dispersion

Guides de conception des systèmes de communication à fibres optiques –

Partie 3: Calcul de la dispersion en mode de polarisation

International Electrotechnical Commission, 3, rue de Varembé, PO Box 131, CH-1211 Geneva 20, Switzerland
Telephone: +41 22 919 02 11 Telefax: +41 22 919 03 00 E-mail: inmail@iec.ch Web: www.iec.ch

Commission Electrotechnique Internationale
International Electrotechnical Commission
Международная Электротехническая Комиссия

CODE PRIX
PRICE CODE W

Pour prix, voir catalogue en vigueur
For price, see current catalogue

CONTENTS

INTERNATIONAL ELECTROTECHNICAL COMMISSION

____

FIBRE OPTIC COMMUNICATION SYSTEM DESIGN GUIDES –

Part 3: Calculation of polarization mode dispersion

FOREWORD

1) The IEC (International Electrotechnical Commission) is a worldwide organization for standardization comprising all national electrotechnical committees (IEC National Committees). The object of the IEC is to promote international co-operation on all questions concerning standardization in the electrical and electronic fields. To this end and in addition to other activities, the IEC publishes International Standards. Their preparation is entrusted to technical committees; any IEC National Committee interested in the subject dealt with may participate in this preparatory work. International, governmental and non-governmental organizations liaising with the IEC also participate in this preparation. The IEC collaborates closely with the International Organization for Standardization (ISO) in accordance with conditions determined by agreement between the two organizations.
2) The formal decisions or agreements of the IEC on technical matters express, as nearly as possible, an international consensus of opinion on the relevant subjects since each technical committee has representation from all interested National Committees.
3) The documents produced have the form of recommendations for international use and are published in the form of standards, technical specifications, technical reports or guides and they are accepted by the National Committees in that sense.
4) In order to promote international unification, IEC National Committees undertake to apply IEC International Standards transparently to the maximum extent possible in their national and regional standards. Any divergence between the IEC Standard and the corresponding national or regional standard shall be clearly indicated in the latter.
5) The IEC provides no marking procedure to indicate its approval and cannot be rendered responsible for any equipment declared to be in conformity with one of its standards.
6) Attention is drawn to the possibility that some of the elements of this technical report may be the subject of patent rights. The IEC shall not be held responsible for identifying any or all such patent rights.

The main task of IEC technical committees is to prepare International Standards. However, a technical committee may propose the publication of a technical report when it has collected data of a different kind from that which is normally published as an International Standard, for example "state of the art".

Technical reports do not necessarily have to be reviewed until the data they provide are considered to be no longer valid or useful by the maintenance team.

IEC 61282-3, which is a technical report, has been prepared by subcommittee 86C: Fibre optic systems and active devices, of IEC technical committee 86: Fibre optics.

The text of this technical report is based on the following documents:

Enquiry draft	Report on voting
86C/296/DTR	86C/346/RVC

Full information on the voting for the approval of this technical report can be found in the report on voting indicated in the above table.

Annexes A, B, C, D and E are for information only.

This publication has been drafted in accordance with the ISO/IEC Directives, Part 3.

This document, which is purely informative, is not to be regarded as an International Standard.

The committee has decided that the contents of this publication will remain unchanged until 2006 At this date, the publication will be

- reconfirmed;
- withdrawn;
- replaced by a revised edition, or
- amended.

A bilingual version of this publication may be issued at a later date.

INTRODUCTION

Polarization mode dispersion (PMD) is usually described in terms of a differential group delay (DGD), which is the time difference between the principal states of polarization of an optical signal at a particular wavelength and time. PMD in cabled fibres and optical components causes an optical pulse to spread in the time domain, which may impair the performance of a fibre optic telecommunication system, as defined in IEC 61281-1.

FIBRE OPTIC COMMUNICATION SYSTEM DESIGN GUIDES –

Part 3: Calculation of polarization mode dispersion

1 Scope

The purpose of this technical report is to provide guidelines for the calculation of polarization mode dispersion (PMD) in fibre optic systems to accommodate the statistical variation of PMD and differential group delay (DGD) in optical fibre cables and components.

This guideline describes methods for calculating PMD due to optical fibre cables and optical components in an optical link. Example calculations are given to illustrate the methods for calculating total optical link PMD from typical cable and optical component data. The calculations include the statistics of concatenating individual optical fibre cables drawn from a specified distribution. The calculations assume that all components have PMD equal to the maximum specified value.

NOTE The statistical specification of the distribution of the PMD of optical fibre cables is a current work item to amend IEC 60794-3, in SC86A/WG3 [2][1]. The agreements following the last ballot (86A/501/CD) are aligned with the methods given in this technical report.

The calculations described cover first order PMD only. This study of PMD continues to evolve, therefore the material in this technical report may be modified in the future. The following subject areas are currently beyond the scope of this technical report, but remain under study:

– calculation of second and higher order PMD;
– accommodation of components with polarization dependent loss (PDL) – if it is assumed that PDL is negligible in optical fibre cables;
– system impairments (power penalty) due to PMD;
– interaction with chromatic dispersion and other nonlinear effects.

Measurement of PMD is beyond the scope of this technical report. Guidelines on the measurement of PMD of optical fibre and cable are given in IEC 61941. The measurement of optical amplifier PMD will be documented in IEC 61290-11-1[2]. The measurement of component PMD will be documented in IEC 61300-3-32[3].

2 Basic design models for total system PMD performance

2.1 Notation

For cabled fibre and components with randomly varying DGD, the PMD frequency domain measurement is based on averaging the individual DGD values for a range of wavelengths. The probability density function of DGD values is known to be Maxwell for fibre, and is assumed to be Maxwell for random components. The single parameter for the Maxwell distribution scales with the PMD value.

[1] Figures in brackets refer to the bibliography.

[2] To be published

[3] To be published

For long fibre and cable (typically longer than 500 m to 1000 m), the PMD value is divided by the square root of the length to obtain the PMD coefficient. For components, the PMD value is reported without normalization. The following terms and meanings will be used to distinguish the various expressions:

– DGD value The differential group delay at a time and wavelength (ps)
– PMD value The wavelength average of DGD values (ps)
– PMD coefficient The length normalized PMD (ps/sqrt(km))
– DGD coefficient The length normalized DGD (ps/sqrt(km))

NOTE The term "DGD coefficient" is used only in some of the calculations. The physical square root length dependence of the PMD value does not apply to DGD.

Deterministic components are those for which the DGD may vary with wavelength, but not appreciably with time. The variation in wavelength may be complex, depending on the number and characteristics of the sub-components within. For these types of components, either the maximum DGD is reported or the wavelength average is reported as the PMD value.

2.2 Calculation of system PMD

PMD values of randomly varying elements can be added in quadrature. Annex A shows the basis of this, as well as one basis for concluding that the Maxwell distribution is appropriate to describe the distribution of DGD values. Annex A describes the concatenation in terms of the addition of rotated polarization dispersion vectors (pdv) which are, for randomly varying components, assumed to be random in magnitude and direction over both time and wavelength.

For deterministic components, the evolution of the pdv with wavelength may be quite complex, but for each wavelength, there is a value that does not vary appreciably with time. Analysis of the relationships in annex A shows that if all deterministic components are at the end of the system and all their pdvs are aligned, the total contribution to the link DGD at a particular wavelength is equal to the sum of the individual DGD values of each deterministic component. The worst case contribution across all wavelengths is therefore the sum of maximum DGD values.

For randomly varying components such as fibre, the statistics of DGD variation imply that there is little wavelength dependence of the PMD value. This leads to an equivalence between PMD measurement methods such as Jones Matrix Eigenanalysis (JME) and interferometric methods (IT) where the wavelength ranges of the two are different. For deterministic elements, there can be distinct dependence of both the DGD and PMD on the wavelength range. Therefore for these elements, the wavelength range must be specified. When doing calculations which combine both randomly varying and deterministic elements, the combined values are only representative of the wavelength overlap.

The relationships of annex A also show an analysis for a more realistic assumption: the deterministic components are embedded within the system and randomly aligned. For this assumption, the DGD values are time randomized across the wavelengths by the downstream fibre. Furthermore, the random alignment of these components with respect to the other elements leads to the following conclusions for embedded deterministic components.

– The quadrature addition of PMD values can be used to calculate the contribution to system PMD.
– The Maxwell distribution can conservatively be used to describe the variation in DGD across time and wavelength.

The following two subclauses provide equations to calculate: a) the maximum PMD value for the system, b) the maximum DGD value for the system. In both cases, the maximum is defined in terms of a probability level that takes into account the statistics of the concatenation of individual cables drawn from a specified distribution of optical fibre cable. For maximum DGD, these statistics are combined with the Maxwell statistics of DGD variation. Clause 3 provides methods of calculating the relevant statistics for the contribution of optical fibre cable, which are used in combination with the component values below.

2.2.1 System maximum PMD

The total maximum PMD value of a fibre optic system including optical fibre cable and other optical components is given by one of the following, depending on the placement of deterministic components:

$$PMD_{\text{tot}} = \left[L_{\text{link}} PMD_Q^2 + \sum_i PMD_{Ci}^2 \right]^{1/2} + \sum_j PMD_{Dj} \tag{1a}$$

$$PMD_{\text{tot}} = \left[L_{\text{link}} PMD_Q^2 + \sum_i PMD_{Ci}^2 + \sum_j PMD_{Dj}^2 \right]^{1/2} + PMD_{D\text{last}} \tag{1b}$$

where

PMD_{tot} is the total system PMD value (ps);

PMD_Q is the link design value of the concatenated optical fibre cable (ps/√km);

L_{link} is the link length (km);

PMD_{Ci} is the PMD value of the i^{th} randomly varying optical component (ps);

PMD_{Dj} is the PMD value of the j^{th} deterministic optical component;

$PMD_{D\text{last}}$ is the PMD value of the last non-embedded deterministic component.

The link design value, PMD_Q, (see 3.1) defines a maximum in terms of the probability, Q, for links with at least M individual cable sections.

NOTE The PMD_Q parameter is not related to the Q factor used in bit error ratio calculations.

The validity of these equations has been demonstrated empirically for systems composed of concatenated optical fibre cables [2]. Equation (1a) is relevant assuming that all deterministic components are at the end of the system. Equation (1b) is relevant assuming that most deterministic components are embedded.

2.2.2 Calculation of system maximum DGD

The total maximum DGD value of a fibre optic system including optical fibre cable and other optical components is given by one of the following, depending on the placement of deterministic components:

$$DGD\max_{\text{tot}} = \left[DGD\max_F^2 + S^2 \sum_i PMD_{Ci}^2 \right]^{1/2} + \sum_j DGD\max_{Dj} \tag{2a}$$

$$DGD\max_{\text{tot}} = \left[DGD\max_F^2 + S^2 \left(\sum_i PMD_{Ci}^2 + \sum_j PMD_{Dj}^2 \right) \right]^{1/2} + DGD\max_{D\text{last}} \tag{2b}$$

where

$DGD\text{max}_{\text{tot}}$	is the maximum system DGD (ps);
$DGD\text{max}_F$	is the maximum concatenated optical fibre cable DGD (ps) (see below);
S	is the Maxwell adjustment factor (see below);
PMD_{Ci}	is the PMD value of the i^{th} random component (ps);
$DGD\text{max}_{Dj}$	is the maximum DGD of the j^{th} deterministic component (ps);
PMD_{Dj}	is the PMD value of the j^{th} embedded deterministic component (ps);
$DGD\text{max}_{\text{Dlast}}$	is the maximum DGD of the last non-embedded deterministic component (ps).

The maximum DGD for optical fibre cable (see 3.2) is defined by a probability, P_F, and reference length. It is computed from the convolution of the distribution of the concatenated link PMD distribution and the Maxwell distribution of DGD values.

For components, the S parameter relates to the probability, P_C, that a random component DGD value exceeds $S \cdot PMD_C$, assuming the Maxwell distribution. The following table shows the relationship of S to probability when the PMD value is defined as the wavelength average.

Table 1 – Probability based on wavelength average

S	Probability
3,0	4,2E-05
3,1	2,0E-05
3,2	9,2E-06
3,3	4,1E-06
3,4	1,8E-06
3,5	7,7E-07
3,6	3,2E-07
3,7	1,3E-07
3,775	6,5E-08
3,8	5,1E-08
3,9	2,0E-08
4,0	7,4E-09
4,1	2,7E-09
4,2	9,6E-10
4,3	3,3E-10
4,4	1,1E-10
4,5	3,7E-11

Annex B shows that the probability that a system DGD value, DGD_{tot}, exceeds $DGD\text{max}_{tot}$ is bounded by the sum of the two probabilities as:

$$P\left(DGD_{\text{tot}} > DGD\max{}_{\text{tot}}\right) \leq P_F + P_C \qquad (3)$$

NOTE The notation $P(\,)$ indicates a probability statement relative to the inequality within the parenthesis.

The above equations are applicable to all links with length less than the reference length. An adjustment for longer lengths is included in 3.2. Equation (2a) is relevant for the assumption that all deterministic components are aligned and at the end of the system. Equation (2b) is relevant for the assumption that almost all deterministic components are randomly aligned and embedded in the system. The multiplication of the deterministic PMD values with the

S parameter treats these elements as though their DGD values are distributed as Maxwell – a conservative assumption that allows the quadrature addition. Because the Maxwell approximation for deterministic elements is conservative, if equation (2a) yields a $DGD\text{max}_{\text{tot}}$ value less than equation (2b), then equation (2a) value should be used (see annex E and [10]).

NOTE 1 The assumption of quadrature addition of DGD values of cabled fibre and randomly varying optical components is subject to experimental verification.

NOTE 2 While it is possible to combine the statistical distributions of random components with cabled fibre, it would require access to information that may not be generally available to any single vendor or customer.

NOTE 3 The DGD specified for deterministic components is assumed to be the maximum across the relevant wavelength range and environmental conditions

Equation (3) illustrates that the total probability of exceeding some overall maximum can be bounded by an addition that does not depend on the relative magnitude of $DGD\text{max}_F$ and $S{\cdot}PMD_C$. Given an overall probability target, one approach is to allocate half the overall allowed probability to fibre and half to components. Annex C provides a worked example for both equations (2a) and (2b).

3 Calculation of cabled fibre PMD

PMD is a stochastic attribute that varies in magnitude randomly over time and wavelength. The variation in the DGD value is described by a Maxwell probability density function that can be characterized by a single parameter, the PMD value (see equation (15) in 3.2.1). This parameter may be the average of the DGD values measured across a wavelength band, or it may be the rms value of these DGD values, depending on the definition chosen. For mode coupled fibre, the PMD coefficient is the PMD value divided by the square root of length.

In accordance with ballot 86A/501/CD, the PMD of cabled fibre should be specified/characterized on a statistical basis, not on an individual fibre basis. Two methods for this specification are proposed: method 1 can be used to obtain PMD_Q, used in 2.2.1, and method 2 can be used to obtain $DGD\text{max}_F$ and P_F, used in 2.2.2. The method and specification values chosen shall be agreed upon between the buyer and the cable manufacturer. Paragraph 3.3 shows how specification values for each method can be selected so the two methods are nearly equivalent.

Method 1 relies on the fact that the mean PMD coefficient of an optical link is the root mean square (quadrature average) of the mean PMD coefficients of the cabled fibres comprising the link. Method 2 assumes the same relationship.

Let x_i and L_i be the PMD coefficient (ps/√km) and length, respectively, of a fibre in the i^{th} cable in a concatenated link of N cables. The PMD coefficient, x_N (ps/√km), of this link is:

$$x_N = \left[\frac{\sum_{i=1}^{N} L_i x_i^2}{\sum_{i=1}^{N} L_i}\right]^{1/2} = \left[\frac{1}{L_{\text{Link}}}\sum_{i=1}^{N} L_i x_i^2\right]^{1/2} \qquad (4)$$

If one assumes that all cable section lengths are less than some common value, L_{Cab}, and simultaneously reducing the number of assumed cable sections to $M = L_{\text{Link}}/L_{\text{Cab}}$, then, for a link comprised of equal-length cables, $L_i = L_{\text{cable}}$, equation (4) becomes

$$x_N \le x_M = \left[\frac{L_{\text{Cab}}}{L_{\text{Link}}} \sum_{i=1}^{M} x_i^2 \right]^{1/2} = \left[\frac{1}{M} \sum_{i=1}^{M} x_i^2 \right]^{1/2} \quad (5)$$

The variation in the concatenated link PMD coefficient, x_M, will be less than the variation in the individual cable sections, x_i, because of the averaging of the concatenated fibres.

Method 1 should be used with equation (1) of 2.2.1. In method 1, the manufacturer supplies a maximum PMD link design value, PMD_{Q}, that serves as a statistical upper bound for the PMD coefficient of the concatenated fibres comprising an optical cable link. For this case, the upper bound for the PMD value of the concatenation of optical fibre cables, $PMD_{F\text{Tot}}$, in equation (1) becomes:

$$PMD_{F\text{Tot}} = PMD_Q \sqrt{L_{\text{Link}}} \quad (6)$$

Unless otherwise specified in the detail specification, the PMD link design value shall be less than 0,5 ps/√km, and the probability that a PMD coefficient of a link comprised of at least 20 cables will exceed the link design value shall be less than 10^{-4}. The link design value shall be computed using a method agreed upon between the buyer and cable manufacturer (see 3.1 for examples).

Because method 1 provides a statistical upper bound on the PMD of concatenated links, approved PMD measurement methods can be used on installed cable links to determine whether their PMD complies with the statistical upper bound stated by the manufacturer. Furthermore, the upper bound can be used to compute the effect of the link PMD on the performance of any type of transmission system and is a more realistic indication of the maximum PMD likely to be encountered in a concatenated link than the value that would be obtained using a worst-case PMD value.

Method 2 should be used with equations (2) and (3) of 2.2.2. Method 2 combines the PMD density function of the concatenated links with the Maxwell probability density function of DGD values to compute an estimate of the probability that the DGD of a concatenated link at a given wavelength exceeds a specified value for a defined reference link.

The specification is that the probability that the DGD over the link exceeds a given value, $DGD\text{max}_F$, shall be less than some maximum, P_F. One useful reference system consists of a concatenated link of 400 km comprised of forty 10 km cable sections. For such a link, the buyer and cable manufacturer may agree on specifying values such as $DGD\text{max}_F$ = 25 ps for $P_F \le 6{,}5 \cdot 10^{-8}$. The particular statistical methodology for their calculation shall be agreed between the buyer and cable manufacturer (see 3.2).

NOTE Subclause 3.3 shows conditions under which the specifications of the two methods are nearly equivalent.

3.1 Method 1: Calculating PMD_Q, the PMD link design value

3.1.1 Determining the probability distribution of the link PMD coefficients

Equation (5) shows that the PMD coefficient, x_M, of a particular concatenated link can be derived from the PMD coefficients of the individual cable sections, x_i, comprising that link. The probability distribution of the link PMD coefficients depends on the distribution of the cable PMD coefficients and the number of cable sections comprising the link.

The following paragraphs describe three methods that can be used to estimate the distribution of the link PMD coefficients. One method is numerical [1] and two are analytic [4]. Of the two analytic methods, the first assumes a specific analytic function for the distribution of the cable PMD coefficients, while the second method makes no such assumptions but invokes an extension of the central limit theorem.

3.1.1.1 Monte Carlo numeric method [1]

The Monte Carlo method can be used to determine the probability density, f_{link}, of the concatenated link PMD coefficients without making any assumption about its functional form. This method simulates the process of building links by sampling the measured cable population repeatedly. PMD coefficients are measured on a sufficiently large number of cabled fibres so as to characterize the underlying distribution. This data is then used to compute the PMD coefficient for a single fibre-path in a concatenated link.

Computation of the link PMD coefficient is made by randomly selecting M values from the measured cabled PMD coefficients, and adding them on an rms basis (in quadrature) according to equation (5). The computed link PMD coefficient is placed in a table or a histogram of values derived from other random samplings. The process is repeated until a sufficient number of link PMD values has been computed to produce a high density (0,001 ps/√km) histogram of the concatenated link PMD coefficient distribution. If used directly, without any additional characterization, the number of resamples should be at least 100 000.

Because of the central limit theorem, the histogram of link PMD coefficients will tend to converge to distributions that can be described with a minimum of two parameters. Hence, the histogram can be fit to a parametric distribution that enables extrapolation to probability levels that are smaller than what would be implied by the sample size. The two parameters will invariably represent two aspects of the distributions: the central value and the variability about the central value. A choice of probability distributions can be made on the basis of the shape of the histogram. Typical distributions could include lognormal (the log of the link PMD coefficients is Gaussian) or one that is derived from the Gamma distribution.

3.1.1.2 Gamma distribution analytic method [4]

The Gamma family of distributions can often be used to represent the distributions of both the measured cable PMD coefficients and the link PMD coefficients. If one assumes that the square of the measured cable PMD coefficients, x_i, is distributed as a Gamma random variable, the probability density of the cabled PMD coefficients is given by

$$f_{\text{cable}}(x;\alpha,\beta) = \frac{2\beta^{\alpha} x^{2\alpha-1}}{\Gamma(\alpha)} \exp\left(-\beta x^2\right) \qquad (7)$$

where x is a possible value of the cable PMD coefficient, $\Gamma(\,)$ is the Gamma function, and the two parameters α and β control the shape of the density. Standard fitting techniques, such as the method of maximum likelihood, can be used to fit equation (7) to measured cable PMD data to find values for α and β.

The probability density of the link PMD coefficients, x_M, of M concatenated equal cable lengths has the same form as equation (7), but with α and β replaced by $M\alpha$ and $M\beta$:

$$f_{\text{Link}}(x;\alpha,\beta,M)=\frac{2(M\beta)^{M\alpha}x^{2M\alpha-1}}{\Gamma(M\alpha)}\exp\left(-M\beta x^2\right) \qquad (8)$$

Consequently, the α and β parameters found by fitting equation (7) to the measured cable PMD coefficients can be used in equation (8) to describe the probability density of the link PMD coefficients.

3.1.1.3 Model-independent analytic method [4]

A more general alternative to the one described in 3.1.1.2 can be used that does not make any assumptions regarding the form of the density function that describes the measured PMD coefficients of the cabled fibre.

After measuring the PMD coefficients, x_i, on N cabled fibres, compute the mean, variance and third moment of their squares

$$\mu_1=\frac{1}{N}\sum_{i=1}^{N}x_i^2 \quad \mu_2=\frac{1}{N-1}\sum_{i=1}^{N}\left(x_i^2-\mu_1\right)^2 \quad \mu_3=\frac{1}{N-1}\sum_{i=1}^{N}\left(x_i^2-\mu_1\right)^3 \qquad (9)$$

Let x_M be a random variable representing the link PMD coefficient of a fibre-path formed from the concatenation of M equal-length cables, and let u be a possible value of x_M. Invoking the extended central limit theorem [5], it can be shown that the distribution of the link PMD coefficients is approximated by:

$$f_{\text{Link}}(u;M)=\Phi[z(u)]+\frac{t}{M^{1/2}}\phi[z(u)]\left[1-z(u)^2\right] \qquad (10)$$

where

$$\phi(z)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right) \qquad \Phi(z)=\int_{-\infty}^{z}\phi(y)dy \qquad (10a)$$

and

$$z(u)=\left(\frac{M}{\mu_2}\right)^{1/2}\left(u^2-\mu_1\right) \quad t=\frac{\mu_3}{6\mu_2^{3/2}} \qquad (10b)$$

Differentiating equation (10) with respect to u provides an approximation to the link PMD coefficient probability density function.

3.1.2 Determining the value of PMD_Q

The density functions found for the link PMD coefficients using one of the three methods described in 3.1.1 are now be used to compute the PMD link design value. For a concatenated link comprised of M cables, the PMD link design value, PMD_Q, is defined as the value that the link PMD coefficient, x_M, exceeds with probability Q:

$$P\left(x_M > PMD_Q\right)=Q \qquad (11a)$$

It follows, that for $N > M$, the probability that x_N exceeds PMD_Q is less than Q:

$$P\left(x_N > PMD_Q\right)<Q \qquad (11b)$$

For discussion purposes, an assumption is made that $N \geq 20$ (the link contains at least 20 cables) and that $Q = 10^{-4}$ (the probability that the link PMD exceeds the PMD design value is less than 0,0001). However, the actual values for M and Q shall be agreed upon between the buyer and seller. The following subclauses discuss how PMD_Q can be found using the cable PMD density functions obtained in 3.1.1.

3.1.2.1 Determining the PMD link design value from the Monte Carlo density of 3.1.1.1

To obtain probability levels of $Q = 10^{-4}$ using a pure numeric approach requires Monte Carlo simulations of at least 10^5 samples. Once this is complete, PMD_Q can be interpolated from the associated cumulative probability density function.

Alternatively, the histogram of the link PMD coefficients can be fit with a parametric distribution to enable extrapolation to lower probability levels than the measurement resampling would otherwise allow. A choice of probability distributions can be made on the basis of the shape of the histogram. Typical distributions could include lognormal (the log of the link PMD coefficients is Gaussian) or one that is derived from the Gamma distribution. After the function is fit, the value for PMD_Q at the Q^{th} quantile can be computed.

3.1.2.2 Determining the PMD link design value from the Gamma density of 3.1.1.2

An excellent approximation for the link PMD coefficient, x_Q, for M cables at the 10^{-4} quantile is given by:

$$PMD_Q = \frac{2{,}004 + 0{,}975\sqrt{M\alpha}}{\sqrt{M\beta}} \qquad (12)$$

where the α and β parameters were those found in 3.1.1.2.

For the 288 randomly selected scaled cabled fibres reported in [6], α = 0,979 and β = 48,6, and PMD_Q = 0,20 ps/√km.

3.1.2.3 Determining the PMD link design value using the model-independent method of 3.1.1.3

The moments computed in 3.1.1.3 can be used to compute the link PMD coefficient, x_Q. For a link comprised of M cables, x_Q at the Q^{th} quantile can be approximated by [5]:

$$PMD_Q = \left[\mu_1 + z_Q\left(\frac{\mu_2}{M}\right)^{1/2} + \frac{\mu_3}{6M\mu_2}\left(z_Q^2 - 1\right)\right]^{1/2} \qquad (13)$$

where z_Q is the Q^{th} quantile of the standard normal distribution. For $N > M = 20$ cables and $Q=10^{-4}$, z_Q = 3,72, the PMD design value becomes:

$$PMD_Q = \left[\mu_1 + 0{,}832\sqrt{\mu_2} + 0{,}107\frac{\mu_3}{\mu_2}\right]^{1/2} \qquad (14)$$

For the 288 randomly selected scaled cabled fibres reported in [6],

$$\mu_1 = 2{,}2 \cdot 10^{-2} \qquad \mu_2 = 7{,}43 \cdot 10^{-4} \qquad \mu_3 = 8{,}26 \cdot 10^{-5}$$

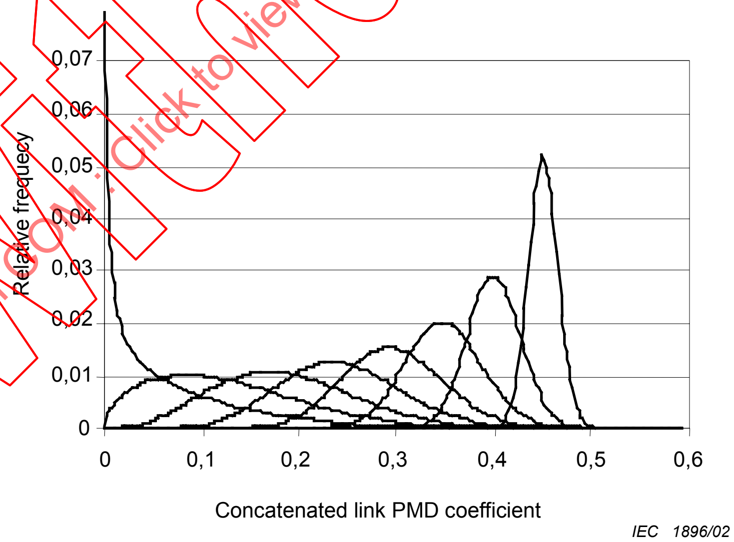
and equation (14) produces PMD_Q = 0,23 ps/√km.

3.2 Method 2: Calculating the probability of exceeding DGD_{max}

PMD induced impairment of an optical signal occurs when the DGD at the signal's wavelength is too high. Since DGD varies randomly with time and wavelength, some means of imposing an upper limit, defined in terms of a low probability value, is necessary for system design. This upper limit is usually associated with a receiver sensitivity penalty. The probability can be associated with a potential PMD-induced impairment time (min/year/circuit). See annex D.

One means of calculating an upper limit on DGD is to multiply the upper limit on the PMD value by a Maxwell adjustment factor, i.e. 3 (see table 1 of 2.2.2). This could also be done with the upper limit represented by PMD_Q. When this is done, one is in effect, assuming that the bulk of the distribution is very close to the upper limit. In reality, the bulk of the distribution is usually well away from the upper limit. Method 2 is intended to provide metrics and methods to take this into account. Because method 2 takes into account the statistics of the individual optical fibre cables and their concatenation, as well as the combined statistics of DGD variation, the values calculated for system design (equation (2b)) are substantially reduced from the "worst-case" values – both in the value of maximum DGD and the probability of exceeding it.

The following figure shows several distributions of concatenated link PMD coefficient. Each distribution just passes the default criteria of $M = 20$, $Q = 10^{-4}$, and $PMD_Q \leq 0{,}5$ ps/√km.



NOTE The above distributions are representative of the Gamma type distribution defined in 3.1.1.2.

Figure 1 – Various passing distributions

The leftmost distributions should provide better DGD performance than the rightmost distribution. Method 2 assigns value to producing a distribution that is more to the left. 3.3 provides a means to link method 1 and method 2 so that, for most practical situations, passing the default method 1 criterion will imply passing a default method 2 criterion.

3.2.1 Combining link and Maxwell variations

DGD coefficient (ps/√km) values, X_M, vary randomly with time and wavelength according to the Maxwell probability density function:

$$f_{\text{Max}}(X_M; x_M) = 2\left(\frac{4}{\pi x_M^2}\right)^{3/2} \frac{X_M^2}{\Gamma(3/2)} \exp\left[-\frac{4}{\pi}\left(\frac{X_M}{x_M}\right)^2\right] \tag{15}$$

where x_M is the PMD coefficient of a concatenated link comprised of M cables as given by equations (4) or (5). The distribution of DGD values (over the length) is obtained by multiplying the DGD coefficient values with the square root of the link length.

To combine the variations in the concatenated link PMD coefficient with the Maxwell variation into a value to be used in a system design, a reference link is defined. Performance on the reference link can then be generalized to other links. The reference link is defined with two parameters, the overall link length, L_{REF}, and the cable section length, L_{Cab}, which is assumed to be constant for all cable sections.

Let $f_{\text{Link}}(xi)$ be the discretized probability density function (histogram) of the values of the concatenated link PMD coefficient values defined by the analysis of the distribution of the measured PMD coefficient values and equation (4). Any of the methods of 3.1.1 for determining the probability density function of the concatenated link can be used.

Let X_{max} be some DGD coefficient value (ps/√km) that is to be used in system design. The probability, P_F, of exceeding X_{max} is:

$$P_F = \sum_i f_{\text{Link}}(x_i)\left[1 - \int_0^{X_{\text{max}}} f_{\text{Max}}(Y; x_i)\,dY\right] = \sum_i f_{\text{Link}}(x_i)\left[1 - \int_0^{\frac{4}{\pi}\left(\frac{X_M}{x_M}\right)^2} \frac{y^{3/2-1}}{\Gamma(3/2)} \exp(-y)\,dy\right] \tag{16}$$

NOTE The rightmost integral is just the standard gamma function.

The maximum DGD over the link derived from optical fibre cable, $DGD\text{max}_F$, is the product of the square root of the reference link length and X_{Max}:

$$DGD\,\text{max}_F = X_{\text{Max}}\sqrt{L_{\text{REF}}} \tag{17}$$

For method 1, the probability is pre-set and the associated PMD_Q value is calculated and required to be less than a specified value. For method 2, $DGD\text{max}_F$ is pre-set and the probability value, P_F is calculated and required to be less than a specified value.

For link lengths less than the reference length, the DGD and probability relationship will be conservative as long as either the installed lengths are less than L_{Cab} or the cable lengths measured to obtain the distribution are less than L_{Cab}. The reduction in averaging because of the reduced number of cable lengths is offset by the decrease in overall length.

For link lengths greater than the reference length, the maximum DGD to optical fibre cable should be adjusted as:

$$DGDadj_F = DGD\max_F \sqrt{\frac{L_{Link}}{L_{REF}}} \qquad (18)$$

3.2.2 Convolution: Theory of method 2

The calculation principle is derived from extending the worst case approach. With this approach, the link PMD distribution is assumed to be a dirac function and the DGD distribution is represented as a Maxwell distribution. The probability that the Maxwell distribution exceeds $DGD\max_F$ yields P_F. These distributions are represented in figure 2.

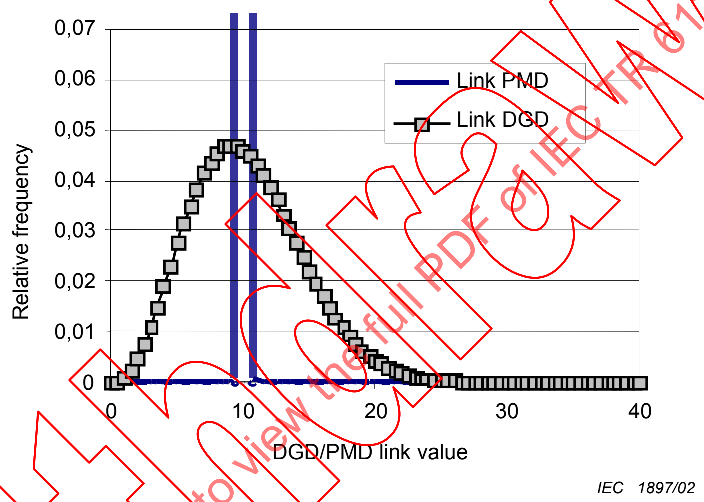


Figure 2 – Worst case approach assumption

NOTE Though not shown, the dirac function illustrated in figure 2 extends to a relative frequency value of 1,0.

Suppose the link PMD distribution could be represented by two dirac functions, each with a magnitude of 0,5. This would represent a situation where half the links were at one value and the other half at another value. The DGD probability density function of the combined distribution would be the weighted total of the two individual Maxwell distributions. Figure 3 illustrates this case.

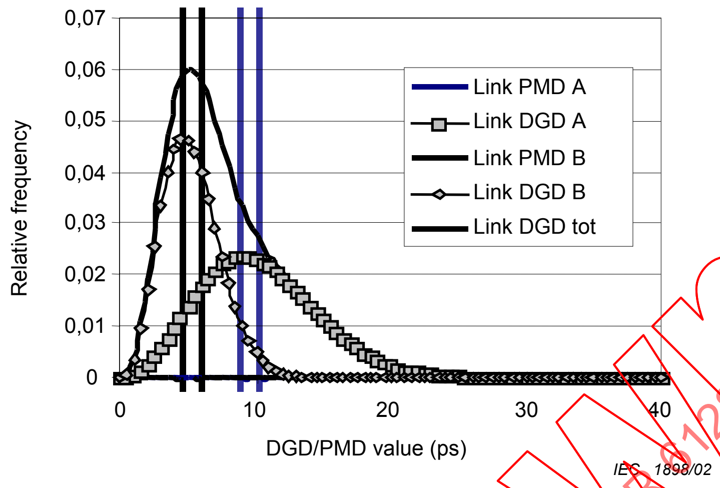


Figure 3 – Convolution of two diracs

NOTE Though not shown, the two dirac functions representing PMD value distributions in figure 3 extend to relative frequency values of 0,5.

In this example, the probability of DGD exceeding 30 ps is reduced by just a little less than a factor of two, compared to the result associated with figure 2.

Full convolution extends this notion to a complete distribution of link PMD coefficients. For the Monte Carlo technique, the histogram of link PMD coefficients may be thought of as a collection of dirac functions. For the continuous models, the probability density function is reduced to a histogram by integrating the curve over the region that is represented as a single histogram bin. The probability that $DGD\max_F$ is exceeded is calculated for each of the histogram bins (using the bin maximum). The weighted total yields P_F.

3.3 Equivalence of methods

Method 1 might be considered most practical for commercial specification because it can be interpreted in terms of the defined measurements. Method 2 provides the most direct information on the possible signal impairments. This subclause shows how the two methods can be compared, and establishes near equivalence of the default specifications.

The method for determining equivalence of statistical criteria relies on a parametric model and is based on the following.

– A process can be characterized by parameters relevant to an assumed parametric distribution type.

– Given these parameters, any statistical criterion can be evaluated to determine whether the process distribution is conforming or not.

– For each criterion, the mathematical space of all possible parameters can be segmented into two regions: conforming and not.

– The boundary between the two regions will form a curve, or envelope, in at least two dimensions. Parameter values falling on one side of the envelope are conforming. Those on the other side are not.

– Processes that are on the conforming side of the envelopes of two criteria pass both criteria. Processes that are on the non-conforming side of the envelopes of two criteria fail both criteria.

– Criteria that pass and/or fail the same process distributions are considered equivalent. In this case, the two envelopes will overlay one another.

3.3.1 Equivalence of the default specifications

Figure 4 shows the envelopes for the default specifications for the two methods, based on the Gamma distribution as defined in 3.1.1.2 and illustrated in figure 1. The x-axis represents the quadrature average of the process ($=\sqrt{\alpha / \beta}$). The y-axis represents a sort of standard deviation metric for the Gamma distribution.

The default criteria are:

Method 1	**Method 2**		
M	20	L_{REF}	400 km
Q	10^{-4}	L_{Cab}	10 km
$PMD_Q \leq 0{,}5$ ps/√km	$DGD\max_F$		25 ps
$P_F \leq 6{,}5 \cdot 10^{-8}$			

Each envelope of figure 4 is built by looping through the possible values of the overall process quadrature average. For each possible value, the β parameter is varied to find the value that just passes the relevant specification. A plot of the relationship of the overall process quadrature average versus $1/2\sqrt{\beta}$ yields the envelope. Processes for which the parameters fall below the envelope pass the specification. Processes for which the parameters fall above the envelope fail.

For the region of most practical interest – where the overall process quadrature average is less then 0,2 ps/√km – if the process passes the default method 1 specification, it is passing the default method 2 criterion. Hence, the method 2 parameters can safely be used in the formulas of 2.2.

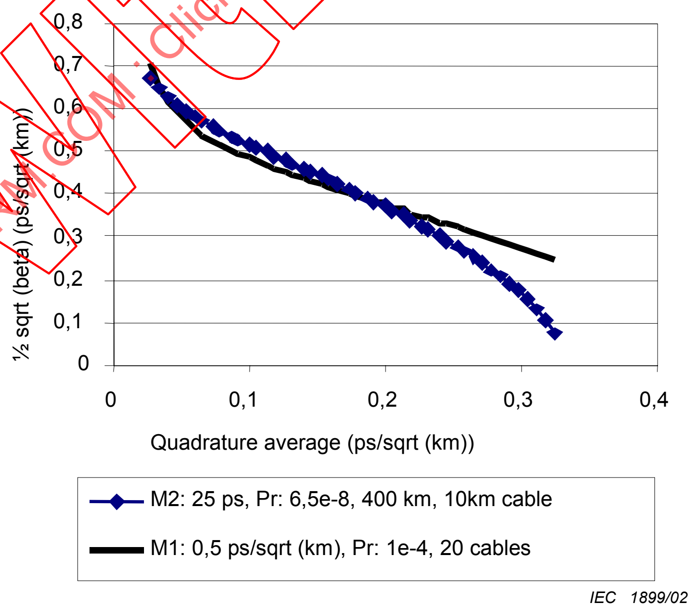


Figure 4 – Equivalence envelopes for method 1/2 defaults

3.3.2 Discussion regarding the basis of the default specifications for method 2

Equation 3 of 2.2 shows that the overall probability that the combined DGD of optical fibre cable and components exceeds $DGD\text{max}_{\text{Tot}}$ is contained by the sum of the probabilities: P_F and P_C. If these probabilities are both set to 6,5 · 10^{-8}, their sum is 1,3 · 10^{-7}, a value that should provide an appropriately low potential PMD induced impairment time (see annex D).

The default specification for method 1 was agreed on the basis of a combination of factors including an analysis of the draft ITU-T Recommendation G.691[4].

The default specification for method 2 was derived so that:

- The above probability objective was met,
- Near equivalence with the default method 1 specification was achieved,
- $DGD\text{max}_F$ is low enough to allow practical system designs. (See annex C for a worked example.)

3.3.3 Calculation of the parameters of figure 4

This calculation for the parameters of the Gamma-type distribution defined in 3.1.1.2 is called the method of moments. Because it is based on the assumption that $M\alpha > 5$, it is best to use it in conjunction with the Monte Carlo method defined in 3.1.1.1.

Define x_I to be one of N computed concatenated link PMD coefficient values based on M cables per link. The quadrature average, v, is calculated as:

$$v = \left(\frac{\alpha}{\beta}\right)^{1/2} = \left(\frac{1}{N}\sum_{i=1}^{N} x_i^2\right)^{1/2} \quad (19)$$

The parameter relating to the standard deviation is calculated as:

$$\frac{1}{2\sqrt{M\beta}} = \left[\frac{1}{N-1}\sum_{i=1}^{N}(x_i - v)^2\right]^{1/2} \quad (20)$$

Multiplying the result of equation (20) by $\sqrt{M}$ will yield the value to be plotted on the y-axis of figure 4. One can also easily compute the individual values of α and β using the values of equations (19) and (20). Since the result is based on the Monte Carlo, the assumption $M\alpha > 5$ can always be met by increasing M.

4 Calculation of optical component PMD

Optical components such as dispersion accommodation devices or optical fibre amplifiers also have small PMD. These devices are characterized as random or deterministic. The distinction is primarily based on the behavior of the curve of DGD versus wavelength and how this curve varies with time.

For random components, small changes in temperature will produce random variations in the DGD curve versus time. When the collection of DGD values are plotted with a histogram, the histogram will follow the Maxwell distribution. Some dispersion accommodation devices have been characterized as random [7].

[4] ITU-T Recommendation G.691 (10/2000), *Optical interfaces for single-channel STM-64, STM-256 and other SDH systems with optical amplifiers*

For deterministic components, small changes in temperature will not produce random variations in the relationship of DGD versus wavelength. Some optical fibre amplifier devices have been characterized as deterministic [8].

Some deterministic components are simple in that they typically include a few individual optical elements and the DGD versus wavelength curve is either flat or has a simple sinusoidal shape. Some deterministic components are complex in that they might be comprised of several individual optical elements and the DGD curve versus wavelength is irregular. For these components, a histogram of the DGD values will begin to approach a Maxwell distribution.

For both types of component, there are at least two metrics of particular interest:

- PMD value: the average of DGD values across a specified wavelength range;
- *DGD*max: the maximum DGD across a specified wavelength range.

The usual form of specification is just a maximum value, although statistical treatments, such as those defined for optical fibre cable, could be defined for component PMD. The number and types of components would make standardization difficult.

The concatenation formulas are given in 2.2.1, equation (1), but are repeated here for completeness.

4.1 Calculation for random components

The PMD values for n random components are given as c_i. The calculated PMD value of the concatenation, c_{Tot} is:

$$c_{\mathrm{Tot}} = \left(\sum_{i=1}^{n} c_i^2 \right)^{1/2} \tag{21}$$

The concatenated PMD value of the random components can be added to the overall PMD value of optical fibre cable as a quadrature total.

Annex B shows that the maximum DGD (defined in terms of probability) of the concatenated random components can be added to the maximum DGD of the optical fibre cable with quadrature total. The combined probability that the total DGD exceeds the computed value is less than the sum of the two probabilities. The maximum DGD for components can be determined by multiplying c_{Tot} by an S value from table 1 of 2.2 using a specified probability.

For the purpose of design calculation, the specified maxima of several types of random component can also be combined using equation (21).

4.2 Calculation for deterministic components

4.2.1 Worse case calculation

Annex A includes one formula as appropriate for worst case assumptions, which are as follows:

- the polarization dispersion vectors are aligned;
- the deterministic components are combined at the end of the link.

For these assumptions, values of the maximum DGD of n components are defined as D_i. The total of the deterministic components, D_{Tot} is calculated as:

$$D_{\mathrm{Tot}} = \sum_{i=1}^{n} D_i \tag{22}$$

This value can be combined with the overall maximum DGD of optical fibre cable and random components by linear addition. Since the values are given as fixed maximums, there is no impact to probability.

Specified maximum DGD values can be substituted into equation (22).

4.2.2 Calculation for embedded deterministic components

This case is particularly suitable for complex components such as optical fibre amplifiers which might consist of several internal optical elements. Annex A shows that the distribution of the DGD values with wavelength approaches a Maxwell distribution. It also shows that when an optical element such as an optical fibre cable follows the deterministic element, the polarization dispersion vector, hence the DGD added by the component, is randomized.

The PMD values of embedded deterministic components can be combined with those of random components according to the formulas of 4.1. Because of the multiplication of the combination by the Maxwell adjustment factor, S, the result will be conservative. For non-embedded deterministic values or for simple deterministic components, where Maxwell variation of DGD with wavelength is not apparent, the linear addition of 4.2.1 can be used without affecting the probability calculations.

Annex E shows an experiment for concatenating some deterministic components (optical fibre amplifiers) with random components (optical fibre). For these experiments, the quadrature total of the PMD values of the individuals is the most accurate predictor of the actual concatenated PMD value.

5 Summary of acronyms and symbols

Table 2 contains a list of the acronyms and definitions used in the body of this technical report. Table 3 contains a list of symbols and the clause in which they are defined.

Table 2 – Acronyms and definitions

Acronym	Definition
PMD	Polarization mode dispersion
DGD	Differential group delay
pdv	Polarization dispersion vector
PMD_Q	Link design value
$DGD\mathrm{max}_F$	Maximum DGD induced by optical fibre cable
$DGD\mathrm{max}_{\mathrm{Tot}}$	Maximum DGD of the link

Table 3 – Symbols and clause of definition

Symbol	Defining clause
PMD_{Tot}	2.2.1
PMD_Q	2.2.1
Q	2.2.1
L_{Link}	2.2.1
PMD_{Ci}	2.2.1
PMD_{Dj}	2.2.1
PMD_{Dlast}	2.2.1
M	2.2.1
$DGD\mathrm{max}_{\mathrm{tot}}$	2.2.2
$DGD\mathrm{max}_F$	2.2.2
$DGD\mathrm{max}_{Dj}$	2.2.2
$DGD\mathrm{max}_{\mathrm{Dlast}}$	2.2.2
P_F	2.2.2
P_C	2.2.2
S	2.2.2
X_N	3
X_M	3
L_i	3
L_{Cab}	3
$PMD_{F\mathrm{Tot}}$	3
f_{cable}	3.1.1.2
f_{Link}	3.1.1.2
α, β, Γ	3.1.1.2
$\mu 1,\ \mu 2,\ \mu 3,\ \Phi,\ \varphi,\ z$	3.1.1.3
X_{Max}	3.2.1
$DGD\mathrm{adj}_F$	3.2.1
L_{REF}	3.2.1
v	3.3.3
c_{Tot}	4.1
D_{Tot}	4.2.1

Annex A
(informative)

PMD concatenation fundamentals

This annex will describe the mathematics for concatenating fibre optic elements (fibre or components) in terms of the polarization dispersion vector, as described by Foschini and Poole [9]. The fundamental relationships are defined, followed by developments related to concatenation of first random then deterministic elements.

A.1 Definitions

The polarization dispersion vector, Ω, and birefringence vector, W, are defined by the evolution of the output stokes vector, s, as a function of position within the element, z, and optical angular frequency, ω. For any position and frequency, there is a rotation matrix, R, also a function of position and frequency, that maps an input stokes vector, s_0, to the output as:

$$s = Rs_0 \tag{A.1}$$

Using the notation, R^T, for the transpose (and inverse) of R, the derivatives of s with respect to position and frequency, assuming that s_0 does not depend on frequency, are given as:

$$\frac{ds}{dz} = \frac{dR}{dz} R^T s = W \times s \tag{A.2}$$

$$\frac{ds}{d\omega} = \frac{dR}{d\omega} R^T s = \Omega \times s \tag{A.3}$$

These equations define W and Ω. The following additional relationships are useful:

$$\frac{d\Omega}{dz} = \frac{dW}{d\omega} + W \times \Omega \tag{A.4}$$

$$\Delta\tau = \|\Omega\| \tag{A.5}$$

Equation (A.4) can be derived from equations (A.2) and (A.3). Equation (A.5) gives the DGD as the length of the polarization dispersion vector.

For W fixed over position, the rotation matrix is defined in terms of a unit rotation vector, y, and angular displacement, γ, as:

$$y = \frac{W}{\|W\|} \qquad \gamma = z\|W\| \tag{A.6a}$$

$$R = yy^T(1-\cos(\gamma)) + I\cos(\gamma) + [y\times]\sin(\gamma) \tag{A.6b}$$

where I is the identity matrix and $[y\times]$ is the matrix that completes the cross product operation.

Furthermore, for this case, the pdv can be written in terms of W and various cross products. Let

$$f^2 = W \cdot W \qquad P = W \cdot \frac{dW}{d\omega} \tag{A.6c}$$

$$\Omega = \frac{1}{f^2}\left[zPW + (1-\cos(fz))W \times \frac{dW}{d\omega} + \frac{\sin(fz)}{f}\left(f^2 \frac{dW}{d\omega} - PW\right)\right] \tag{A.6d}$$

A.2 Concatenation – General

The concatenation of two elements, A through B, can be given in terms of the initial and final stokes vectors as:

$$s_{AB} = R_B R_A s_0 \tag{A.7}$$

The derivative of (A.7) is

$$\frac{ds_{AB}}{d\omega} = \frac{dR_B}{d\omega} R_A s_0 + R_B \frac{dR_A}{d\omega} s_0 \tag{A.8a}$$

$$= \frac{dR_B}{d\omega} R_B^T R_B R_A s_0 + R_B \frac{dR_A}{d\omega} R_A^T R_B^T R_B R_A s_0 \tag{A.8b}$$

$$= \Omega_B \times s_{AB} + R_B\left(\Omega_A \times R_B^T\right) s_{AB} \tag{A.8c}$$

$$= (\Omega_B + R_B \Omega_A) \times s_{AB} \tag{A.8d}$$

Since (A.8d) is in the form of the definition of the pdv, the pdv for AB, Ω_{AB}, is:

$$\Omega_{AB} = \Omega_B + R_B \Omega_A \tag{A.9}$$

Equation (A.9) can be extended through an arbitrary number of optical elements by recursive application of rotation on the last output and addition.

A.3 Application to random elements

Foschini and Poole [9] concluded that for long enough lengths of optical fibre over which mode coupling is occurring, the contents of the polarization dispersion vector can be described as Gaussian identically distributed independent random variables. Extension of equation A.9 leads to the conclusion that the pdv of the concatenation will also have the same distribution, but with increased variance due to the addition of several random vectors. (The random rotation of a Gaussian random vector is a Gaussian random vector with the same variance.) In particular, define the variance of the vector element values of the ith pdv as $\sigma_i^2/3$. The variance of the vector element values of the concatenated pdv are given as:

$$\sigma_{\text{tot}}^2 = \sum_i \sigma_i^2/3 \tag{A.10}$$

This is related to the PMD (rms) value as:

$$PMD_{\text{tot}} = \left\langle \Delta\tau^2 \right\rangle^{1/2} = \left\langle \Omega \bullet \Omega \right\rangle^{1/2} = \left(3\sigma_{\text{tot}}^2\right)^{1/2} = \left(\sum_i \sigma_i^2\right)^{1/2} \tag{A.11}$$

The same relationship occurs on the individual sections with $PMD_i = \sigma_i$ so the quadrature addition of the PMD values of individual random elements is justified for the concatenated PMD value. If one assumes that the PMD of the several elements are equal and that their lengths are also equal, (A11) yields the square root dependence of PMD on overall length – hence the square root length normalization used for the PMD coefficient.

The DGD values are the length of the pdv. If the vector element values are Gaussian independent identically distributed random values, the DGD is the square root of the sum of squares of three Gaussians. This is the definition of the Maxwell distribution, which is also the square root of a chi-square random variable with three degrees of freedom.

A.4 Application to deterministic elements

If the values of the birefringence vector, W, of several deterministic components are aligned just right, the sum of the pdv of the concatenation will just be the linear sum of the individual pdvs. In this case, the components are said to be aligned.

Analysis of equation (9) for the case of these components following a series of random elements leads to adding the PMD of the random elements to the linear addition of the deterministic components. Similarly, the worst case random DGD is added to the sum of worst case deterministic DGD values.

When the pdvs of the deterministic elements are randomly aligned, the rotations will become random and the length of the concatenated pdv will form a distribution. This distribution will be representative of the DGD values across wavelengths and concatenations, but for a given concatenation of deterministic elements, the particular values are not expected to change with time. To evaluate this distribution, a series Monte Carlos were completed. The different Monte Carlos each represent the concatenation of a different number of randomly oriented vectors of a common length. The vector length, L_v, is given in terms of the number of vectors concatenated, n_v, as:

$$L_v = \frac{1}{\sqrt{n_v}} \tag{A.12}$$

The quadrature sum of the individual pdv values is therefore one and the worst case orientation value is equal to $\sqrt{n_v}$. This allows overlaying the distributions formed by adding different numbers of components and a comparison with a Maxwell distribution from a random component with a PMD value of one. Figure A.1 shows the result laid out as a series of histograms. Figure A.2 shows the cumulative probability in the tail of the distributions.

For all but the Maxwell distribution, the cumulative probabilities are bounded by $\sqrt{n_v}$, as expected. In addition, for all cases, the Maxwell distribution extends to larger values than the rotated vector result. This is the sense in which the Maxwell distribution is conservative with regard to estimating the distribution of randomly oriented but fixed length vectors.

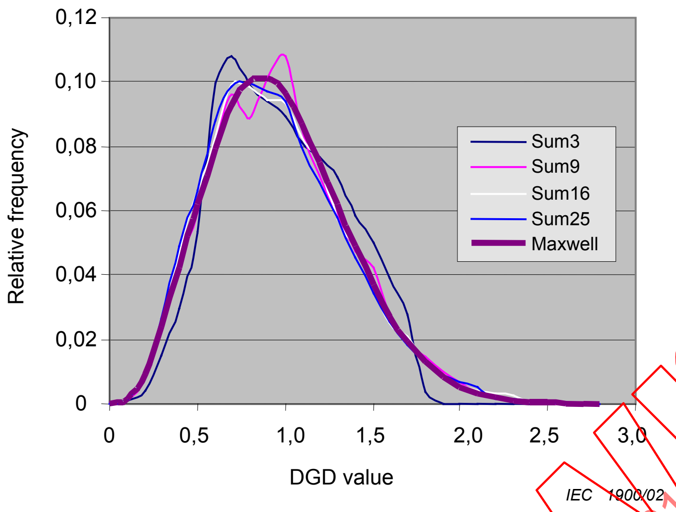


Figure A.1 – Sum of randomly rotated elements

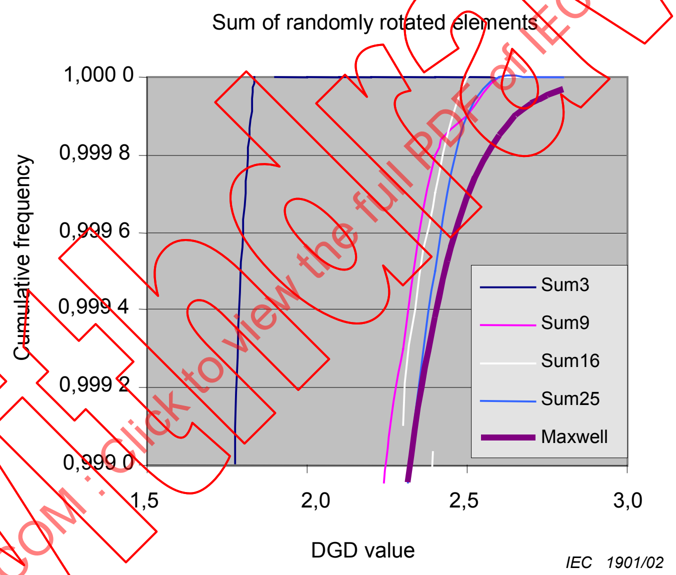


Figure A.2 – Sum of randomly rotated elements

So the application of the Maxwell distribution in connection with the PMD value will yield a maximum estimated DGD that is larger than actual for a concatenation of randomly oriented deterministic elements of equal PMD. Since, for most link designs, the value of the deterministic components is assumed to be the maximum specified value, this equal size assumption is also conservative.

For a particular orientation on a particular link, the concatenation of deterministic components will be the same over time and this might be a concern. Examination of equation (9) yields a surprising result in this regard. If a deterministic element is followed by a fibre or other random component, the rotation (R_B) of that element is applied to the pdv of the deterministic component. Since this rotation is random, the effect is to randomize the effect of the deterministic element over both time and wavelength. The overall contribution is then not fixed and it can be considered as just another random optical element – with the exception that the probability estimates will be higher than actual.

For embedded deterministic components, the probability treatment that is applicable for random components is therefore appropriate for deterministic components.

Annex B
(informative)

Combining Maxwell extrema from two populations

The purpose of this annex is to demonstrate the validity of equations (2a) and (3) of clause 2. Equation (2a) combines the DGD extrema of two populations, optical fibre cable and random components, to obtain a combined extreme value. The two populations are each characterized by probability levels associated with the separate extrema. Inequality 3 asserts that the probability that a DGD value of the concatenation of the two populations exceeds the combined extreme value is bounded by the sum of the two separate probabilities.

The analysis shows that only two assumptions are needed:

- The distribution of DGD is Maxwell-like in that only one parameter describes it.
- The parameter of the concatenation is given as the quadrature total of the fibre and component parameters.

This demonstration is done in four steps:

- define the Maxwell distribution;
- define a convolution;
- define a double convolution;
- valuate the characteristics of a double convolution.

B.1 Maxwell distribution definitions

The probability that a random DGD value, $\Delta\tau$, exceeds a given maximum DGD, D, depends on the PMD value, d, according to:

$$P(\Delta\tau > D; d) = 1 - \int_0^D 2\left(\frac{4}{\pi d^2}\right)^{3/2} \frac{x^2}{\Gamma(3/2)} \exp\left[-\frac{4}{\pi}\left(\frac{x}{d}\right)^2\right] dx \qquad \text{(B.1a)}$$

To simplify notation, a function, $P_{Max}(S)$, is defined as:

$$P_{\text{Max}}(S) = 1 - \int_0^S 2\left(\frac{4}{\pi}\right)^{3/2} \frac{x^2}{\Gamma(3/2)} \exp\left[-\frac{4}{\pi} x^2\right] dx \qquad \text{(B.1b)}$$

Then $$P(\Delta\tau > D; d) = P_{\text{Max}}\left(\frac{D}{d}\right) \qquad \text{(B.1c)}$$

NOTE $P_{\text{Max}}(S)$ is monotonically decreasing in S.

B.2 Convolution definition

This describes the convolution of the Maxwell distribution with a distribution of PMD values. The resultant distribution is not Maxwell, but does have the characteristic of describing the distribution of DGD values across all possible PMD values. The definition is done in terms of a discretized distribution, or histogram, of PMD values for simplicity. Since the granularity of the histogram is arbitrary and in practice, defined by the measurement resolution, the simplification, relative to using integral expressions, is warranted.

The discretized distribution of PMD values is described in terms of a histogram with the "bin" PMD values given as d_i and relative frequency given as p_i for the i^{th} bin. The combined probability that a given maximum DGD value, D, is exceeded is:

$$P(\Delta\tau > D) = \sum_i p_i P_{\max}\left(\frac{D}{d_i}\right) \quad \text{(B.2)}$$

B.3 Convolution of optical fibre cable and random components

This clause outlines the notation for the double convolution of cables and random components. For optical fibre cable, the distribution of PMD values is that of the concatenated link, as derived from clause 3. For random components, a distributional notation will be maintained for generality, but in practice, the specification values of all components will be combined in quadrature to form one value. The component distribution is then a histogram with only one non-zero bin.

Let f_i and p_{fi} represent the histogram of optical fibre cable PMD values. Let c_j and p_{cj} represent the histogram of component PMD values. The two distributions are assumed to be independent. Given a fibre PMD value, f, and a component PMD value, c, the PMD value of the concatenation, d, is assumed to follow:

$$d = \left(f^2 + c^2\right)^{1/2} \quad \text{(B.3)}$$

Further assume that the extreme values and probability limits are provided separately for optical fibre cable and components as:

$$P(\Delta\tau > F) = \sum_i p_{fi} P_{\max}\left(\frac{F}{f_i}\right) < P_F \text{ for optical fibre cable} \quad \text{(B.4a)}$$

$$P(\Delta\tau > C) = \sum_j p_{cj} P_{\max}\left(\frac{C}{c_j}\right) < P_C \text{ for components} \quad \text{(B.4b)}$$

The problem is to determine the probability limit, P_D, for the concatenation of cable and components relative to a maximum, D, given as:

$$D = \left(F^2 + C^2\right)^{1/2} \quad \text{(B.5)}$$

If there were access to the full distributions of both cable and components, this probability could be calculated as:

$$P_D = P(\Delta\tau > D) = \sum_i \sum_j p_{fi} p_{cj} P_{\max}\left[\frac{\left(F^2 + C^2\right)^{1/2}}{\left(f_i^2 + c_j^2\right)^{1/2}}\right] \tag{B.6}$$

Equation (B.6) is the double convolution of cable and components.

B.4 Evaluation of the double convolution

This subclause will demonstrate the following relationship for the problem defined in B.3.

$$P_D < P_F + P_D \tag{B.7}$$

The formulas and values for cables and components are given separately from equations (B.4a) and (B.4b). These are combined and rewritten as:

$$P_F + P_C \geq \sum_i p_{fi} P_{\max}\left(\frac{F}{f_i}\right) + \sum_j p_{cj} P_{\max}\left(\frac{C}{c_j}\right) = \sum_i \sum_j p_{fi} p_{cj}\left[P_{\max}\left(\frac{F}{f_i}\right) + P_{\max}\left(\frac{C}{c_j}\right)\right] \tag{B.8}$$

The last expression in (B.8) is due to the definition of the histogram probabilities: $1 = \sum_i p_{fi} = \sum_j p_{cj}$.

The asserted inequality, (B.7), can now be written as:

$$P_D = \sum_i \sum_j p_{fi} p_{cj} P_{\max}\left[\frac{\left(F^2 + C^2\right)^{1/2}}{\left(f_i^2 + c_j^2\right)^{1/2}}\right] \leq \sum_i \sum_j p_{fi} p_{cj}\left[P_{\max}\left(\frac{F}{f_i}\right) + P_{\max}\left(\frac{C}{c_j}\right)\right] \leq P_F + P_C \tag{B.9}$$

Inequality B.9 will be true if, for each i, j pair, the following is true:

$$P_{\max}\left[\frac{\left(F^2 + C^2\right)^{1/2}}{\left(f_i^2 + c_j^2\right)^{1/2}}\right] \leq P_{\max}\left(\frac{F}{f_i}\right) + P_{\max}\left(\frac{C}{c_j}\right) \tag{B.10}$$

Inequality (B.10) will be true if the left side is smaller than either of the terms on the right side. Since $P_{\max}(S)$ is decreasing in S, this will be true if either of the following two inequalities is true:

$$\frac{F^2 + C^2}{f_i^2 + c_j^2} \geq \frac{F^2}{f_i^2} \tag{B.11a}$$

or

$$\frac{F^2 + C^2}{f_i^2 + c_j^2} \geq \frac{C^2}{c_j^2} \tag{B.11b}$$

(B.11a) is true when:

$$\frac{F^2 + C^2}{f_i^2 + c_j^2} = \left(\frac{F}{f_i}\right)^2 \frac{1 + (C/F)^2}{1 + (c_j/f_i)^2} \geq \frac{F^2}{f_i^2} \quad \text{(B.12)}$$

which occurs when

$$\frac{1 + (C/F)^2}{1 + (c_j/f_i)^2} \geq 1 \quad => \quad \frac{C}{c_j} \geq \frac{F}{f_i} \quad \text{(B.13a)}$$

Similarly, inequality (B.11b) is true when:

$$\frac{C}{c_j} \leq \frac{F}{f_i} \quad \text{(B.13b)}$$

Since either (B.13a) is true or (B.13b) is true, then either (B.11a) is true or (B.11b) is true. This implies that (B.10) is true for each instance of i and j. The inequality of (B.7) is therefore verified.

In the context of equation (2a) from clause 2, the component distribution has only one non-zero probability value, c, given as:

$$c = \left(\sum_i PMD_{Ci}^2\right)^{1/2} \quad \text{and} \quad C = Sc \quad \text{(B.14)}$$

This implies

$$P_C = P_{\max}\left(\frac{C}{c}\right) = P_{\max}(S) \quad \text{(B.15)}$$

for optical fibre cable set F, in the above notation, to $DGD\max_F$ that is used in clause 2. P_F is the same in this annex as in clause 2.